Ex-Post Unbalanced Tournaments

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Abstract
Tournaments create strong incentives under the assumption that the competition between the agents is balanced. If, at the outset, one agent is stronger than the other, the tournament is ex-ante unbalanced and incentives break down. Handicaps can in this case restore incentives. In practice, competing agents are often overall equally strong but have different sorts of strengths. Then competition will typically be unbalanced ex-post and incentives break down, but handicaps cannot be used. We show how a simple means, introducing the possibility of a tie, can often resolve the problem. We examine under what conditions incentives are maximized by ties.

Key words: contracts; limited liability; moral hazard; subjective evaluation; tie.
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1 Introduction

In tournaments, players compete for given prizes that are distributed according to relative performance.\footnote{The theory of tournaments and contests builds on the seminal articles by Tullock (1980), Lazear and Rosen (1981) and Nalebuff and Stiglitz (1983). More recent work comprises Marino and Zábojník (2004), Feess et al. (2008), Gershkov et al. (2009), Parreiras and Rubinchik (2010), and Suzuki (2012).} The best performing player gets the highest prize, the second-best the second highest prize and so on. There are many examples of tournaments in real life: In sports contests, individual athletes or teams try to win against other individuals or teams, respectively. Many firms use tournaments to determine bonus payments for workers and managers. In job-promotion tournaments, workers compete for a more attractive position at the next hierarchy level. Politicians participate in election contests to run for higher-level offices. The main reasons for the widespread use of tournaments are that they induce high efforts, are simple to implement and that they can be applied even when other incentive schemes do not work (e.g., under subjective or unverifiable performance measurement; Malcomson 1984, 1986).

A key requirement is that the competition in tournaments must be balanced to avoid discouragement of all players (Lazear and Rosen 1981, Nalebuff and Stiglitz 1983, O’Keefe et al. 1984). Tournament competition can be unbalanced in two ways. On the one hand, individual players may have a competitive advantage known at the beginning of the tournament (e.g., a higher productivity). Such ex-ante unbalanced competition can be easily cleared by an appropriate seeding of players or the introduction of handicaps (Lazear and Rosen 1981, Nalebuff and Stiglitz 1983). On the other hand, it is also possible that players start in a fair tournament but, when a winner is selected, it turns out that one player has obtained an unpredictable competitive advantage so that we can speak of ex-post unbalanced competition. In that case, the aforementioned solutions – seeding or handicaps – do not work, because the advantaged player is not known ex ante.

There exist many tournament situations with ex-post unbalanced competition. For
example, workers with comparable qualifications will usually have different strengths and weaknesses. One of the strengths may turn out to be decisive, but ex ante neither the firm nor the workers know whose strengths will be decisive ex post. More generally, individuals typically differ in many respects, but often it is not clear in advance which specific difference will yield a competitive advantage when completing complex or innovative tasks or selling to customers with unknown preferences. As another example, imagine a sales contest with two competing agents and suppose that there is one big client (that buys a lot of product) and many small clients. The difference in sales between the two agents is likely to be disproportionately influenced by the event which agent will serve the big client. If this event is mainly determined by luck (e.g., which sales agent arrives first), one of the agents will be advantaged ex post, but ex ante it is unclear which agent will be the lucky one. Finally, imagine a situation with subjective performance evaluation by a supervisor and agents that differ in at least one dimension (e.g., age, appearance, gender, or nationality). If the agents expect that the supervisor has some prejudice regarding the dimension(s) in which they differ, again agents might start in a fair tournament but end up in an ex-post unbalanced competition. In all these examples, no one knows ex ante which specific agent will be favored, but if it is sufficiently likely that the winner is mainly determined by a competitive advantage ex post, the incentives of all agents will be harmed.

In such situations, it seems useful to introduce an additional hurdle for the winner to leave the tournament outcome open. In this article, we analyze a two-agent tournament and show under which conditions a minimum distance by which the best performing agent must outperform his opponent is an effective hurdle. If neither agent has won by this minimum distance, a tie will occur. In that case, either the sum of the tournament prizes is equally shared among the agents or a coin toss decides which player obtains the winner prize and which one the loser prize.² Either alternative en-

²Besides the seniority criterion, random selection of candidates in case of a tie is widely used as part of the official selection and promotion rules in the U.S. civil services; see, e.g., the “Rules of the
sures Malcomson’s (1984, 1986) self-commitment property of a tournament, which is important to prevent opportunistic behavior of the tournament organizer – the principal – in case of unverifiable performance measures. Contracts of this type retain much of the simplicity of the standard tournament contract without ties and we show that the possibility of a tie is often beneficial for the principal as a measure against ex-post unbalanced competition. In this case, the principal voluntarily introduces the possibility of a tie even though the monitoring technology identifies a clear tournament winner. Our results can answer questions such as: Should a worker obtain a large winner bonus even if his performance is only slightly better than that of his opponent? In a song contest, should the best singer win by a minimum lead of 1 or 10 merit points?

Our findings show that, if the problem of ex-post unbalanced competition is sufficiently severe and dominates the influence of idiosyncratic noise, the principal will prefer the possibility of a tie. Ex-post unbalanced competition will be a relevant problem if either agent might have a competitive advantage when the tournament winner is selected, if the advantage is not negligible and if this event is sufficiently likely. In this situation, both agents anticipate that the outcome of the tournament will be strongly influenced by the ex ante unknown competitive advantage. Consequently, under the standard contract without tie, both agents exert only minimal effort to save effort costs. Introducing the possibility of a tie is then beneficial for the principal because it makes the agent that benefits from the competitive advantage less likely to win. Suppose the tournament winner gets $w > 0$, the loser gets zero and there is a tie-breaker. Similar tie-breaking rules can also be found in private corporations (e.g., Kim 1995, p. 114). In addition, we observe random election of politicians and judges in case of a tie; see, e.g., Koppel (2012) and Schwarz (2014).

If the sum of paid tournament prizes is not the same under any event, a rational principal will ex post always claim the tournament outcome that minimizes his wage costs. As this opportunistic behavior is anticipated by the agents, incentives would be erased.
no possibility of a tie. If it is likely that one of the agents has a strong competitive advantage ex post, neither agent should exert effort ex ante. This rationale holds for each agent irrespective of whether he or his opponent will receive the advantage. If the principal introduces the possibility of a tie, incentives for both agents will be restored. Now being advantaged does not imply winning, but exerting effort yields a realistic chance of earning $w$. Being disadvantaged does not imply losing and exerting effort yields a realistic chance of earning at least $w/2$.

As an example, recall the case of the two sales agents and the big client from above. In that scenario, the principal would like to filter out the influence of the big client on the outcome of the sales contest. This can be done by defining a minimum distance comparable in size to the amount of the sales to the big client so that the possibility of a tie keeps alive the competition between the sales agents. However, if sufficiently strong idiosyncratic noise leaves the tournament outcome open and, hence, already reduces the impact of the competitive advantage, a possible tie becomes counterproductive. In that situation, the standard tournament contract can be optimal even if it is likely that one of the agents has a competitive advantage ex post.

In practice, the influence of idiosyncratic noise can sometimes be controlled by the principal (e.g., by varying the sample size if the performance measure is based on sampling). In other words, the principal can choose between several monitoring technologies that differ in their precision defined as the inverse of the performance measure’s noise variance (e.g., Holmström 1999). In that case, the principal can combine the minimum distance and the precision of the monitoring technology as instruments against ex-post unbalanced competition. Our results show that the former instrument (i.e., introducing ties) is more effective than the latter one.

To examine the robustness of our findings we discuss several variants of the main model. If agents are ex ante heterogeneous, the principal can combine ties with handicaps by using individualized minimum distances to even out competition. If outputs

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4See the discussion of the monitoring intensity principle in Milgrom and Roberts (1992).
are verifiable, the principal can additionally benefit from ties by saving labor costs. In promotion tournaments, possible ties are useful to reduce the risk of promoting unsuitable agents. If not the principal but a supervisor subjectively evaluates the agents, a possible tie tends to limit the scope for vertical collusion.

Practical examples show that firms indeed use tournaments with a minimum distance or ties. Murphy (1999) and De Angelis and Grinstein (2014) give overviews about top manager compensation in practice. They report that firms often pay CEOs based on relative performance (see also Gibbons and Murphy 1990). Several firms combine relative performance pay with a minimum distance so that CEOs will only earn a bonus if they achieve a threshold relative to their peers. Although CEOs are agents that belong to different principals, the basic structure of these pay schemes is closely related to our contracts with ties. Two other kinds of tournaments – called sum-of-targets approach and discretionary pools – are applied to determine compensation of top managers below CEO rank. Here, a bonus pool is defined for a certain set of managers and the pool is distributed according to individual performance. Performance thresholds exist both on the collective level for the complete bonus pool as well as on the individual management level.

Another example of tournaments with ties is the widely used forced-ranking system, where workers are subjectively rated according to relative performance and then assigned to different grades. The grades determine individual compensation. Boyle (2001) reports that about 25 percent of the so-called Fortune 500 companies use forced-ranking systems (e.g., Cisco Systems, Intel, Sun Microsystems, Conoco). Murphy (1992) gives a detailed description of the forced-ranking system employed at Merck, a large pharmaceutical corporation. The system installed in 1986 has five ranks: 5-8% percent of the workers have to be assigned to the highest rank, 15-17% to rank 2, 65-70% to rank 3, 5-8% to rank 4, and 2% to the lowest rank. The compensation is determined by the rank, so that there is a tie between all workers that are assigned to the same rank.
The article is organized as follows. In Section 2, we introduce the tournament model. A characterization of the optimal tournament contract is given in Section 3. Section 4 addresses an illustrative case where the agents’ individual advantages have a binary distribution. We present necessary and sufficient conditions under which ex-post unbalanced competition is strong enough to dominate noise so that introducing the possibility of a tie is optimal. In Section 5, we compare possible ties and a low precision of the monitoring technology as alternative instruments against ex-post unbalanced competition. In Sections 6 and 7, we derive general sufficient conditions for the optimal tournament contract to include a possible tie or not. Variants of the main model and related work are discussed in Sections 8 and 9. Section 10 concludes.

2 The Model

We consider a variant of the model by Lazear and Rosen (1981), which describes a situation where a principal has to employ two agents in order to run a business. The three players are risk neutral. Agent $i$’s ($i = 1, 2$) unverifiable output for the principal (or the subjective measure of agent $i$’s performance) is described by the function

$$x_i(e_i) = h(e_i) + \eta_i + \sigma \epsilon_i,$$

where $e_i \geq 0$ denotes agent $i$’s effort choice with $h$ as corresponding concave production function that satisfies $h(0) = 0$ and $h'(e_i) > 0$, $h''(e_i) \leq 0$. $\eta_i$ and $\epsilon_i$ are random variables and $\sigma$ is a positive parameter unless stated otherwise. We assume that the distribution of the random variables is known by the principal and the agents. The variable $\epsilon_i$ describes idiosyncratic noise. $\epsilon_1$ and $\epsilon_2$ are assumed to be i.i.d. The parameter $\sigma$ determines the magnitude of the influence of this noise. The larger $\sigma$ the higher will be the impact of pure luck on relative performance $x_i(e_i) - x_j(e_j)$.

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Additive separability of performance measures is a standard assumption in the literature on tournaments and signal jamming, see e.g. Lazear and Rosen (1981), Meyer (1991), Fairburn and Malcomson (2001), Höfler and Sliwka (2003), and Bar-Isaak and Hopner (2014).
We assume $\sigma$ is exogenous, except in Section 5, where $\sigma$ can be influenced by the principal.

The random variable $\eta_i$ introduces the possibility that agent $i$ has an individual advantage or disadvantage ex post compared to the initial situation. Following the examples in the introduction, $\eta_i$ can indicate the effective ability of agent $i$, i.e., the suitability of his qualifications, personal strengths and other individual characteristics for successfully completing a complex task or selling a product to a customer. The complexity of the task or the uncertainty about the customers’ preferences imply that the realization of the random variable $\eta_i$ is not known ex ante. When interpreting $x_i (e_i)$ as the subjectively evaluated performance of agent $i$, we can think of $\eta_i$ as an individual (unconscious) bias of the principal or a supervisor toward agent $i$. The random variable $\Delta \eta := \eta_1 - \eta_2$ denotes the competitive advantage that may exist for one of the agents ex post; that is, $\Delta \eta$ describes the advantage of one agent over the other.

We do not assume that $\eta_1$ and $\eta_2$ are independent. Hence, we allow the individual ex-post advantages or disadvantages of the two agents to systematically differ or tend into the same direction. In the perhaps most interesting applications, $\eta_1$ and $\eta_2$ have the same distribution, but we also allow the possibility that the distributions differ. This possibility includes the case of a degenerate distribution where $\eta_1$ and $\eta_2$ are equal to different constants, which is the case of an unfair tournament as considered, for example, by Schotter and Weigelt (1992). The principal cannot observe any component of the right-hand side of (1) so that we have a typical moral-hazard problem.

We assume $(\eta_1, \eta_2)$ and $(\epsilon_1, \epsilon_2)$ are independent. To condense notation, we define $\Delta \epsilon := \epsilon_1 - \epsilon_2$. We assume that $\Delta \eta + \sigma \Delta \epsilon$ has a continuous density$^6$ $g$ and denote the corresponding cumulative distribution function by $G$.

$^6$A sufficient condition under which $\Delta \eta + \sigma \Delta \epsilon$ has a continuous density is that the noise terms $\epsilon_i$ have a bounded, or more generally, a square-integrable density. This is shown in Lemma 1 in the appendix.
Exerting effort $e_i$ entails costs $c(e_i)$ for agent $i$ with $c(0) = c'(0) = 0$, and $c'(e_i) > 0$ and $c''(e_i) \geq 0$ for $e_i > 0$. We additionally assume that $c'(e_i)/h'(e_i)$ is strictly increasing. Let the agents’ reservation values be normalized to $\bar{u} = 0$. Moreover, each agent is protected by limited liability in the sense that his wage payment must be non-negative. Each agent maximizes expected income, consisting of expected wage payment minus effort costs, whereas the principal maximizes output minus wage payments by implementing a finite effort level.

To analyze a possible solution to the problem of ex-post unbalanced competition between the agents, we consider an extended version of the standard tournament contract that has been introduced by Nalebuff and Stiglitz (1983). They suggest that the better performing agent will only receive the winner prize $w_H$ (and the other agent the loser prize $w_L$), if he has won by a minimum distance $\delta \geq 0$ that has been fixed in advance. If neither agent has won by this minimum distance, there will be a tie and the tournament prizes will be equally shared or distributed via a fair coin toss so that each agent’s (expected) pay is $(w_H + w_L)/2$. The tournament contract is written as $(w_H, w_L, \delta)$ with tournament prizes satisfying the limited-liability condition $w_H, w_L \geq 0$. As Nalebuff and Stiglitz (1983), we focus on non-discriminatory contracts: each agent gets the same contract offer $(w_H, w_L, \delta)$. This assumption is very natural if $\eta_1$ and $\eta_2$ have the same distribution, because agents have identical cost and production functions. The assumption is also justified if the distribution of the $\eta_i$ is not known to outsiders and equal opportunity laws prohibit unequal treatment of equal individuals in practice.

The timeline is the usual one in moral-hazard models. First, the principal offers

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7 As we assume performance measures are unverifiable, the total compensation in case of a tie must be equal to $w_L + w_H$ to ensure Malcomson’s (1984, 1986) self-commitment property. In Section 8 we discuss to what extent our results continue to hold for verifiable performance measures when the payment in case of a tie is flexible.

8 We show in Section 8 that some of our results can be extended to contracts with individual minimum distances $\delta_i$, $i = 1, 2$.

9 See Schotter and Weigelt (1992) on tournaments and equal opportunity laws.
the agents a contract \((w_H, w_L, \delta)\). Then, the agents can accept or reject the contract offer. If the agents accept, they will choose a non-negative effort level. Finally, outputs are realized and the principal and the agents receive their payoffs according to the contract.

3 Characterization of the Optimal Tournament Contract

The game is solved by backwards induction. First, we consider the agents’ effort choices in the tournament for a given contract \((w_H, w_L, \delta)\) under the assumption that both agents participate. Then, we derive the optimal contract \((w^*_H, w^*_L, \delta^*)\) that satisfies the agents’ participation constraints, the incentive constraints and the limited-liability constraints.

At the tournament stage, agent 1 maximizes

\[
EU_1(e_1) = w_L - c(e_1) + \Delta w \cdot P(x_1(e_1) > x_2(e_2) + \delta) \\
+ \frac{\Delta w}{2} \cdot P(|x_1(e_1) - x_2(e_2)| \leq \delta)
\]

with \(\Delta w := w_H - w_L\) denoting the spread between winner and loser prize. In any case, each agent gets at least the loser prize and has to pay his effort costs \(c(e_i)\). If agent \(i\) wins by the minimum distance \(\delta\), he will receive the additional prize spread \(\Delta w\). In case of a tie, each one will obtain the additional amount \(\Delta w/2\). Because

\[
P(x_1(e_1) > x_2(e_2) + \delta) = 1 - G(h(e_2) - h(e_1) + \delta)
\]

and

\[
P(x_2(e_2) > x_1(e_1) + \delta) = G(h(e_2) - h(e_1) - \delta),
\]

agent 1’s objective function can be rewritten as

\[
EU_1(e_1) = w_L + \frac{\Delta w}{2} \left[2 - G(h(e_2) - h(e_1) + \delta) - G(h(e_2) - h(e_1) - \delta)\right] - c(e_1),
\]
and agent 2’s objective function reads as

$$EU_2(e_2) = w_L + \frac{\Delta w}{2} [G(h(e_2) - h(e_1) - \delta) + G(h(e_2) - h(e_1) + \delta)] - c(e_2).$$

We assume that an equilibrium in pure strategies exists at the tournament stage and that it is characterized by the first-order conditions\(^{10}\)

$$\frac{\Delta w}{2} [g(h(e_2) - h(e_1) - \delta) + g(h(e_2) - h(e_1) + \delta)] = \frac{c'(e_1)}{h'(e_1)} = \frac{c'(e_2)}{h'(e_2)},$$

which yields

$$e_1 = e_2 =: e \quad \text{and} \quad \frac{\Delta w}{2} [g(-\delta) + g(\delta)] = \frac{c'(e)}{h'(e)}. \quad (2)$$

Thus, the equilibrium is symmetric, irrespective of any symmetry properties of the stochastic terms \(\eta_1\) and \(\eta_2\).

At the first stage of the game, the principal chooses the optimal contract \((w^*_H, w^*_L, \delta^*)\) that maximizes

$$2h(e) - w_L - w_H \quad (3)$$

subject to the incentive constraint (2), the limited-liability constraint \(w_H, w_L \geq 0\) and the participation constraint \(EU_i(e) \geq 0\). Note that the latter one is always satisfied:\(^{11}\) each agent can ensure himself a non-negative expected utility – and, thus, at least his reservation value – by accepting any feasible contract (with non-negative tournament prizes due to the agents’ limited liability) and choosing zero effort. As a direct consequence, the principal need not pay a positive loser prize to satisfy the participation constraint. Moreover, a positive \(w_L\) would increase the principal’s labor costs \(w_L + w_H\) and decrease the agents’ incentives (see (2)). Hence, we have \(w^*_L = 0\).

The resulting incentive constraint \(w_H [g(-\delta) + g(\delta)] = 2c'(e)/h'(e)\) implies that the optimal minimum distance \(\delta^*\) maximizes \(g(-\delta) + g(\delta)\), because otherwise the principal

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\(^{10}\)The problem that the existence of pure-strategy equilibria cannot be guaranteed in general is well-known in the tournament literature; see, e.g., Lazear and Rosen (1981) and Nalebuff and Stiglitz (1983). See also G"urtler (2011) for a sufficient condition.

\(^{11}\)See, in analogy, for the one-agent case Schmitz (2005b) and Ohlendorf and Schmitz (2012).
could save labor costs $w_H$ by further increasing $g(-\delta) + g(\delta).$ By inserting the incentive constraint
\[ w_H = \frac{2c'(e)}{[g(-\delta^*) + g(\delta^*)]h'(e)} \]
into the principal’s objective function (3), the optimal effort that is implemented by the principal satisfies
\[ e^* \in \arg \max_e 2h(e) - \frac{2c'(e)}{[g(-\delta^*) + g(\delta^*)]h'(e)}. \] (4)

Altogether, the optimal tournament contract is summarized in the following proposition:

**Proposition 1** $\delta^*$ is optimal if and only if it is a mode of $g(\delta) + g(-\delta)$, the density of $|\Delta \eta + \sigma \Delta \epsilon|$. If $\delta^*$ is an optimal minimum distance, the optimal tournament contract is given by $(w^*_H, w^*_L, \delta^*)$ with $w^*_L = 0$ and $w^*_H = 2c'(e^*)/([g(-\delta^*) + g(\delta^*)]h'(e^*))$, where $e^*$ is described by (4).

Proposition 1 shows that the optimal minimum distance is determined by the stochastic interplay of the competitive advantage $\Delta \eta$ and idiosyncratic noise, described by the density $g$. The optimal minimum distance maximizes incentives for every prize spread $\Delta w$, so that the labor costs become as low as possible for every implementable effort level $e$.

In the next sections, we will address in various settings the question under what conditions the introduction of a possible tie strictly improves the standard tournament contract, which specifies only a winner prize and a loser prize. The following two elementary criteria are immediate consequences of (2) and Proposition 1.$^{13}$

**Corollary 1**

a) If $g$ is strictly convex in a neighborhood of 0, then the standard tournament contract can be strictly improved by introducing a possible tie.

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$^{12}$A sufficient condition for the existence of an optimal $\delta^* \in [0, \infty)$ is that the noise terms $\epsilon_i$ have a square-integrable density, see Lemma 1 in the appendix.

$^{13}$For result a) note that the convexity assumption on $g$ implies that the even function $g(\delta) + g(-\delta)$ is strictly convex in a neighborhood of 0. Thus, the function has a strict local minimum at $\delta = 0$ and there exists $\delta^* > 0$ with $g(\delta^*) + g(-\delta^*) > 2g(0)$. 

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b) If $g$ has a global maximum at 0, then the standard tournament contract without tie is optimal.

The conditions in Corollary 1 are formulated in terms of the density of $\Delta \eta + \sigma \Delta \epsilon$, which summarizes the relevant information about the joint distribution of advantages and noise. By contrast, the criteria for the optimality of a contract with or without tie, respectively, that we present later in Propositions 5, 6, and 8 impose separate conditions on advantages, noise, and $\sigma$. The conditions in Corollary 1 are closer to the characterization of the optimal contract in Proposition 1, but it should often be easier to have a good intuition for the conditions presented later.

4 Binary Advantages

We start our analysis with the illustrative case of a binary distribution for the individual advantage $\eta_i$ to identify the main factors that lead to a tournament contract with tie or a standard tournament contract as optimal solution, respectively. For simplicity, idiosyncratic noise is assumed to follow a standard normal distribution.\textsuperscript{14}

**Proposition 2** Suppose $\epsilon_1$ and $\epsilon_2$ are standard normal variables and $\eta_1$ and $\eta_2$ are binary variables with

$$P(\eta_i = a) = p, \quad P(\eta_i = 0) = 1 - p, \quad i = 1, 2,$$

where $a > 0$ and $0 < p < 1$. If $P(\eta_1 \neq \eta_2) \leq \frac{2}{3}$ or $\sigma^2 \geq a^2/2$, then the principal prefers the standard tournament contract to a tournament contract with tie. If $P(\eta_1 \neq \eta_2) > \frac{2}{3}$ and $\sigma > 0$ is so small that

$$\frac{2}{3 - 2e^{-a^2/(4\sigma^2)}} \leq P(\eta_1 \neq \eta_2),$$

then introducing a possible tie is optimal (i.e., $\delta^* > 0$).

\textsuperscript{14}All proofs are relegated to the appendix.
To provide some intuition for Proposition 2 suppose the \( \eta_i \) describe agents’ effective abilities. These abilities can depend on the matching of ex ante unknown aspects of the task and the strengths of the agents, and the match quality determines whether one of the agents has a competitive advantage ex post. In this setting, ex-post unbalanced competition can arise for two different reasons. First, agents’ effective abilities can follow different distributions (e.g., agents have different training qualifications). Second, unbalancedness can occur when the \( \eta_i \) have identical distributions but are not independent. Proposition 2 focuses on this second, perhaps more interesting, kind of unbalanced competition.

Proposition 2 shows that the possibility of a tie will be optimal if (a) the problem of ex-post unbalancedness is sufficiently severe and (b) unbalanced competition is not offset by a large impact of idiosyncratic noise. A possible tie is then beneficial for the principal to counterbalance probably uneven competition.

Condition (a) means that the probability of an ex-post unbalanced competition must be sufficiently large, \( P(\eta_1 \neq \eta_2) > \frac{2}{3} \), and that \( a \) must be sufficiently large, so that inequality (6) is satisfied. Under this condition, an unbalanced tournament between an, ex post, rather strong contestant and a rather weak one is likely, and so a countervailing possible tie is attractive for the principal. Intuitively, as the probability distributions are known by the agents, they take into account whether they face a probably unbalanced competition. In the binary setting, there are four possible cases – both agents have effective ability \( a \), both have ability 0, or agent 1 (agent 2) has ability \( a \) and his opponent has ability 0. If the two unbalanced cases are likely and if \( a \) is large, both agents anticipate that, with a high probability, the outcome of the tournament will mainly be determined by the agents’ effective ability difference. Therefore, in the absence of ties, both agents would optimally choose low efforts to save effort costs.

Condition (b) requires that the problem of ex-post unbalancedness dominates the impact of idiosyncratic noise. Inequality (6) is only satisfied if idiosyncratic noise is not
too large (i.e., $\sigma$ must be sufficiently small). This condition can be explained as follows. Even if agents’ effective abilities differ significantly with high probability, a sufficiently large value of $\sigma$ already eliminates the potentially unbalanced competition, as a strong impact of noise makes winning of a predominant favorite less likely. Because, ex ante, this rationale holds for both agents, equilibrium efforts are high and introducing a possible tie would harm competition.

Condition (a) can only be satisfied if the effective abilities $\eta_1$ and $\eta_2$ are negatively correlated. Suppose, for example, that the principal is a financial service provider employing two agents who both sell fixed-income securities and risky assets. Imagine that one agent is more talented in selling fixed-income securities whereas the other agent is better at selling risky assets. Let both agents give financial advice to a certain number of customers and assume that, at the end of the day, the customers either buy no product or choose one of the products according to the performance of the relevant market index. If it has performed well, the customers tend to prefer risky assets, if it has performed badly, they tend to prefer fixed-income securities. In this case, agents’ outputs are negatively correlated. If one of the agents benefits from the performance of the market index, the other one does not. A similar situation arises in the example with one big client mentioned in the introduction. In that case, $\eta_i = a$ ($\eta_i = 0$) denotes the event that agent $i$ sells (does not sell) to the big client, with $\eta_1$ and $\eta_2$ being negatively correlated. Thus, if one agent acquires the big client, he will have a competitive advantage whereas the other agent will inevitably be disadvantaged.

We now consider an application of Proposition 2 to a model where effective abilities depend explicitly on the matching of agents’ strengths and aspects of the task, which we encapsulate in a random task environment $S$. At the time when the principal chooses a contract and the agents choose their effort levels, the state of the task environment is unknown, only the distribution of $S$ is available (e.g., based on past

\[P(\eta_1 = 1, \eta_2 = a) = 2(p - p^2)(1 - \rho)\]

Thus, if $\rho \geq 0$, then $P(\eta_1 \neq \eta_2) \leq \frac{1}{2}$. See Bar-Isaak and Hörner (2014) on symmetric ability uncertainty with negatively correlated abilities.

15If $\rho$ denotes the correlation coefficient of $\eta_1$ and $\eta_2$, then, by (5), $P(\eta_1 \neq \eta_2) = 2(p - p^2)(1 - \rho)$. Thus, if $\rho \geq 0$, then $P(\eta_1 \neq \eta_2) \leq \frac{1}{2}$. See Bar-Isaak and Hörner (2014) on symmetric ability uncertainty with negatively correlated abilities.
experience). Suppose that, in any particular state, agent $i$ either has or does not have an individual advantage compared to the initial situation. In the first case, $\eta_i = a$, otherwise $\eta_i = 0$. Let $\pi_i(s)$ denote the probability of having an individual advantage in state $s$, that is,

$$\pi_i(s) = P(\eta_i = a|S = s), \quad i = 1, 2.$$  

Suppose further that $\eta_1$ and $\eta_2$ are conditionally independent, given $S$, and that before $S$ becomes known, both agents have the same probability of having an individual advantage:

$$E[\pi_1(S)] = E[\pi_2(S)].$$

Hence, ex ante it is not clear whether one of the agents will have a competitive advantage over the other and, if this happens, which one will be the lucky one. We have $P(\eta_i = a) = E[\pi_1(S)], \ i = 1, 2,$ $P(\eta_1 = \eta_2 = a) = E[\pi_1(S)\pi_2(S)],$ and so the inequality $P(\eta_1 \neq \eta_2) > \frac{2}{3}$ in Proposition 2 becomes

$$E[\pi_1(S)(1 - \pi_2(S))] > \frac{1}{3}. \quad (7)$$

Thus, if this inequality holds, introducing a possible tie is optimal for small $\sigma$; otherwise, the optimal minimum distance is 0 for all $\sigma$, provided noise is Gaussian as before. E.g., in the balanced case where $\pi_1 = \pi_2$, we have $E[\pi_1(S)(1 - \pi_2(S))] \leq \frac{1}{4}$ and not introducing a possible tie is optimal.

We next evaluate condition (7) for a special case. Suppose that, given the state of the task environment, abilities are deterministic. That is, for each agent $i$ there is a region $A_i$ so that $\pi_i(s) = 1$ or 0 according as $s \in A_i$ or not. Suppose also that $P(S \in A_1) = P(S \in A_2)$. Then (7) holds if and only if $P(S \in A_1 \Delta A_2) > \frac{2}{3}$, where $A_1 \Delta A_2$ is the symmetric difference of $A_1$ and $A_2$, which is the set of states in which exactly one agent has an advantage. For example, imagine that the principal is an advertising agency that delegates the development of a new advertising strategy for a business customer to a team of two employees. Let one of the employees be better at creating text and the other one at creating pictures, and assume that, in
advance, it is uncertain whose skills will be most valuable. Denote the type of the underlying business problem by $S$ (i.e., there are business problems that primarily need an excellent text, other problems require excellent pictures, and some require both for a successful advertising campaign). Then the advertising agency should introduce the possibility of a tie to balance competition if and only if it is sufficiently likely that the specific ability of only one employee is advantageous.

5 Precision of the Monitoring Technology

The previous analysis has shown that equilibrium efforts depend on both the minimum distance $\delta$ (i.e., the possibility of a tie) and $\sigma$, the level of noise. In the presence of possible ex-post unbalanced competition, the effect of a large $\sigma$ seems to be similar to that of introducing ties: if the level of noise is high, then an agent considering the possibility of being advantaged cannot be so sure of winning the tournament, and an agent considering the possibility of being disadvantaged may be encouraged by the prospect of luck. Both considerations seem to suggest exerting greater effort than under a low level of noise.

To compare the two effects and their strengths we assume in this section that $\sigma$ is endogenous so that the principal can choose both $\delta$ and $\sigma$.\textsuperscript{16} An endogenous noise level $\sigma$ occurs, for example, if performance measures are based on sampling and the principal can choose the sample size. We disregard the uninteresting case where noise can be eliminated and assume that $\sigma \geq \sigma_0$ for some fixed $\sigma_0 > 0$.\textsuperscript{17} The parameter $\sigma$ can be nicely interpreted in terms of the monitoring technology used by the principal:

\textsuperscript{16}See Chapter 7 in Milgrom and Roberts (1992) for a single-agent moral-hazard model with endogenous variance.

\textsuperscript{17}Lazear and Rosen (1981) and Nalebuff and Stiglitz (1983) emphasize that pure-strategy equilibria can only exist if there is sufficient noise in the competition. That the choice of larger noise can be beneficial when ties are ruled out and agents’ abilities are known has been shown by Kono and Yagi (2008).
1/$\sigma^2$ is proportional to the precision\(^{18}\) of the performance measure $x_i(e_i)$ (see, e.g., Holmström 1999, Höffler and Sliwka 2003, Bar-Isaac and Hörner 2014).

First, we reconsider the binary model and solve for the optimal precision $1/\sigma^2$ given $\delta = 0$. We will then analyze a general setting and allow the principal to choose the optimal mixture of minimum distance and precision. We keep all assumptions as in Sections 2 and 3. To emphasize the dependence on $\sigma$ we now denote the density of $\Delta \eta + \sigma \Delta \epsilon$ by $g_\sigma$. Then the optimal choice of $\sigma$ and $\delta$ is found by maximizing $g_\sigma(\delta) + g_\sigma(-\delta)$ subject to the relevant constraint. The corresponding optimal prizes and the induced efforts are determined as in Proposition 1.

The following proposition presents the optimal precision for the binary model. The solution is given in terms of $W(z)$, the principal branch of the Lambert $W$ function, which is defined as the inverse of the strictly increasing function $w \mapsto we^w$, $w \in [-1, \infty)$.\(^{19}\)

**Proposition 3** Consider the binary model with Gaussian noise of Proposition 2. Suppose only tournament contracts without tie are feasible and the principal can choose the level of noise $\sigma \in [\sigma_0, \infty)$, where $\sigma_0 > 0$.

a) If $P(\eta_1 \neq \eta_2) \leq 1/(1 + 2e^{-3/2})$, then the principal cannot benefit from choosing a level $\sigma > \sigma_0$; more precisely, $\sigma_0$ is the unique optimal level of noise.

b) Suppose $P(\eta_1 \neq \eta_2) > 1/(1 + 2e^{-3/2})$. Then the principal will strictly benefit from choosing a level $\sigma > \sigma_0$ if and only if $\sigma_0 \in (\underline{\sigma}, \bar{\sigma})$, where the interval $(\underline{\sigma}, \bar{\sigma})$ is given as follows:

$$\bar{\sigma} = \left[2 - 4W \left(-\frac{\sqrt{2} P(\eta_1 = \eta_2)}{2 P(\eta_1 \neq \eta_2)}\right)\right]^{-\frac{1}{2}};$$

if $P(\eta_1 \neq \eta_2) < 1$, then $\sigma$ is the unique element of $(0, \bar{\sigma})$ with $g_\sigma(0) = g_\sigma(0)$; and otherwise, $\sigma = 0$. Moreover, if $\sigma_0 \in (\underline{\sigma}, \bar{\sigma})$, then the unique optimal level is $\bar{\sigma}$.

\(^{18}\)The precision of $x_i(e_i)$ is the reciprocal of the conditional variance of $x_i(e_i)$ given agent $i$’s ‘true’ output $h(e_i) + \eta_i$. This conditional variance is equal to the variance of $\sigma \epsilon_i$, which is proportional to $\sigma^2$.

\(^{19}\)See Corless et al. (1996) for various properties of $W(z)$. 

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The proposition shows that if the probability of a competitive advantage $\Delta \eta \neq 0$ is sufficiently large, the principal can benefit from choosing a lower precision to balance competition. The larger the magnitude of the possible advantage, $a$, the lower the optimal precision.

To relate the two approaches to increase incentives recall that, by Proposition 2, the standard tournament contract cannot be improved by introducing the possibility of a tie if $P(\eta_1 \neq \eta_2) \leq \frac{2}{3}$ or $\sigma = \sigma_0 \geq a/\sqrt{2}$. In this case, the principal cannot benefit from choosing $\sigma > \sigma_0$ either. This follows from Proposition 3 by observing that $1/(1 + 2e^{-3/2}) \approx 0.69 > \frac{2}{3}$ and $\bar{\sigma} \leq a/\sqrt{2}$ because $W(z) \leq 0$ for $-e^{-1} \leq z \leq 0$. On the other hand, whenever the principal can increase his profits under the standard tournament contract by choosing some $\sigma > \sigma_0$, it may still be possible to further increase profits by allowing ties and adjusting $\sigma$. We shall prove that such an improvement can always be achieved, and this result is not restricted to the binary model.

We now turn to the two dimensional problem of choosing a noise level $\sigma \geq \sigma_0$ and a minimum distance $\delta \geq 0$. The following proposition shows that if the principal can offer a contract with possible ties, he should not decrease the precision of the performance measures $x_i$, provided the noise has a normal distribution.\textsuperscript{20} The proposition applies to every distribution of $(\eta_1, \eta_2)$.

**Proposition 4** Suppose the principal can choose a level of noise $\sigma \in [\sigma_0, \infty)$, where $\sigma_0 > 0$, and a contract $(w_H, w_L, \delta)$ with any minimum distance $\delta \in [0, \infty)$. Suppose the noise terms $\epsilon_i$ have a normal distribution. Then the profit that the principal obtains under a contract that is optimal for the level $\sigma$ is a strictly decreasing function of $\sigma$. In particular, the unique optimal choice for $\sigma$ is $\sigma_0$.

The result is intuitive in light of Holmström’s informativeness principle (see Holmström (1979)).

\textsuperscript{20}The proof of Proposition 4 shows that the result holds for a wider class of noise distributions, namely the class of all non-degenerate stable distributions, which also includes e.g. all Cauchy distributions.
ström 1979): To avoid a loss of incentives from ex-post unbalancedness, the principal may or may not choose a positive minimum distance, but he will always choose the monitoring technology with the highest precision. Concerning the optimal mixture of minimum distance and precision as a means against ex-post unbalancedness, the principal strictly prefers the minimum distance as the only instrument. In view of Proposition 4 we focus in the rest of the article on the choice of the minimum distance \( \delta \) for any given \( \sigma > 0 \).

6 Sufficient Conditions for the Optimality of a Tie

We now consider general distributions for a possible competitive advantage and for the noise and address the question whether the possibility of a tie yields a strict improvement of the standard tournament contract. This general framework also allows for \( \eta_1 \) and \( \eta_2 \) to follow different distributions. In view of the results for the binary model in Section 4, a simple complete classification seems out of reach. In this section, we present conditions that ensure that a possible tie yields a strict improvement. Conditions under which a standard tournament contract cannot be improved are derived in the next section.

If, in the binary model, a competitive advantage \( \Delta \eta \neq 0 \) is likely to exist, introducing a possible tie is optimal, provided \( \sigma \) is not too large. In the following, we analyze a setting similar to that addressed by Meyer (1991).\(^{21}\) We assume that the \( \eta_i \) are certain not to coincide, specifically, that the support of \( \Delta \eta \) does not contain the origin. This means that there will be one agent that will have at least a minimal competitive advantage. As no two agents are absolutely identical in reality, this assumption does not seem to be too strong. For example, we can think of a situation in which agents’ effective abilities depend on their vocational training and in which

\(^{21}\)Meyer (1991) considers a tournament model with linear production technology and symmetric type uncertainty with \( \Delta \eta \neq 0 \). However, an agent’s production function does not include effort, so that incentives cannot be an issue in her model.
the two agents are hired from different labor pools; e.g., both agents are economics students from different universities that offer different bachelor and master programs. In that case, it is not unrealistic to assume that given a certain task either agent 1 or agent 2 will have at least a marginal competitive advantage depending on the received training. As another example, recall the situation from the introduction where one of the agents might be favored due to an unconscious bias of the supervisor when subjectively evaluating the agents’ performance. For that case, we only assume that there exists at least a marginal relative bias toward one of the agents.

Under the above condition on the competitive advantage $\Delta \eta$, but under an otherwise arbitrary distribution of $\Delta \eta$, and under some mild condition on the noise terms, the introduction of a possible tie turns out to be optimal when $\sigma$ is not too large.\(^{22}\) We prove two versions of this result. The qualitative version, Proposition 5, contains a mild decay condition on the density of $\epsilon_i$. The condition is trivially satisfied for every density with a bounded support. The quantitative version, Proposition 6, gives a lower bound on the optimal minimum distance and specifies exactly when $\sigma$ is small enough.

**Proposition 5** Suppose the support of $\Delta \eta$ does not contain the point 0. Suppose the noise terms $\epsilon_i$ have a density $f$ that satisfies the decay condition

$$\lim_{|x| \to \infty} xf(x) = 0. \quad (9)$$

Then $\delta^* > 0$, i.e. introducing a possible tie is optimal, provided $\sigma > 0$ is sufficiently small.

Proposition 5 shows that one of the main insights of the binary model also holds in a quite general setting: given that there will be an agent who will have at least a marginal competitive advantage ex post, the possibility of a tie will be optimal to fine-tune incentives if the influence of idiosyncratic noise is not too large.

\(^{22}\)For large $\sigma$, the principal prefers the standard tournament contract without tie under every distribution of $\Delta \eta$ and a wide class of noise distributions, see Proposition 8.
Proposition 6 Suppose
\[ P(|\Delta \eta| \geq \alpha) = 1 \]
for some constant \( \alpha > 0 \). Suppose \( \Delta \epsilon \) has a density that, for some \( \beta \geq 0 \), is strictly convex on \((\infty, -\beta]\) and on \([\beta, \infty)\).\(^{23}\) If \( \beta \sigma < \alpha \), then every optimal minimum distance is greater than or equal to \( \alpha - \beta \sigma \).

The lower bound on the optimal minimum distance sheds light on how the size of the distance reflects the degree of ex-post unbalanced competition. If the impact of noise is small, that is, if \( \sigma \) is small, the bound is close to \( \alpha \), the minimal competitive advantage. This finding corresponds nicely to our result for the binary model that a tie is introduced by the principal to counterbalance presumably uneven competition. If \( \beta > 0 \), then, as \( \sigma \) increases from 0 to \( \alpha/\beta \), the lower bound decreases from \( \alpha \) to 0, quantifying the notion that the increasing impact of noise dilutes unbalanced competition and that, also in the general setting, noise and the minimum distance act as substitutes. We will see in the next section that, under mild conditions on the noise, for \( \sigma \) sufficiently large, the optimal minimum distance is zero, so that, in a sense, noise completely eliminates unbalanced competition and the standard tournament contract remains optimal.

7 Sufficient Conditions for the Optimality of the Standard Tournament Contract

Most of the existing tournament models consider a contract that specifies tournament prizes but ignore the possibility of introducing ties. In this section, we derive sufficient conditions under which the standard tournament contract without tie is indeed optimal. The following proposition applies to all \( \sigma \geq 0 \).

\(^{23}\)The convexity assumption is satisfied for most relevant densities with unbounded support, including normal, Laplace and Cauchy densities.
Proposition 7 Set $\theta_i = \eta_i + \sigma \epsilon_i$. The standard tournament contract without tie is optimal under each of the following three conditions:

a) $\theta_1$ and $\theta_2$ are i.i.d.

b) $\theta_1$ and $\theta_2$ are independent and each $\theta_i$ has a unimodal density that is symmetric about 0.

c) $\theta_1$ and $\theta_2$ have a log-concave joint density and $\theta_1 - \theta_2$ has the same distribution as $\theta_2 - \theta_1$.

Proposition 7 identifies several situations where the interplay of the individual advantages, $\eta_i$, and the idiosyncratic noise leads to a density $g$ that has a global maximum at zero, so that, by Corollary 1, the standard tournament contract should not be supplemented by a possible tie. Result a) includes the benchmark tournament model of Lazear and Rosen (1981, Section II) without ex-post unbalancedness, i.e., $P(\eta_i = 0) = 1$ for $i = 1, 2$. In this case, a possible tie would strictly reduce incentives, because in a balanced competition agents become discouraged when they must beat their opponent by a minimum distance. In cases b) and c), incentives even monotonically decrease in the minimum distance $\delta$. The proof shows that in these cases density $g$ is symmetric and unimodal. The larger $\delta$ the more we will go to the tail of $g(-\delta) + g(\delta)$ and, hence, the lower will be an agent’s marginal winning probability when exerting effort in equilibrium. According to incentive constraint (2), this negative incentive effect makes equilibrium efforts monotonically fall in $\delta$.

Proposition 2 has shown for the binary case that introducing a possible tie will not be optimal, if the impact of the idiosyncratic noise is sufficiently large. The following result demonstrates that this finding can be generalized. The proposition applies to a wide class of distributions of the noise terms $\epsilon_i$ and places no condition on the distributions of the individual advantages $\eta_i$.

\footnote{For this finding, see also Eden (2007).}
Proposition 8 Suppose $\epsilon_1$ and $\epsilon_2$ have a common differentiable density $f$ and $f'$ is bounded and integrable. Then for all $\sigma$ sufficiently large, the standard tournament contract without tie is optimal.

If noise is normally distributed and the possible competitive advantage is bounded in the sense that $|\Delta \eta|$ is bounded, we can derive an explicit threshold for $\sigma$ beyond which rejecting a possible tie is optimal. The threshold depends on the magnitude of the possible competitive advantage: the smaller the bound on $|\Delta \eta|$, the smaller the threshold.

Example 1 If $|\Delta \eta|$ is bounded by a constant $C$ and $\epsilon_1$ and $\epsilon_2$ have a standard normal distribution, then the standard tournament contract without tie is optimal, provided $\sigma \geq C/\sqrt{2}$.

8 Discussion

Here we consider several variants of our main model and explore when the principal can benefit from introducing a possible tie in these settings. Except for the specific changes mentioned we keep all other assumptions.

□ Heterogeneous agents. Agents have been assumed to be homogeneous in that they have the same cost and the same production functions. In this section, we sketch the optimal tournament contract with or without tie when agents’ cost functions differ. We continue to assume that both agents have the same production function, but the analysis for individual production functions is similar. Let agent $i$’s effort costs be described by the increasing and convex function $c_i(e_i)$ ($i = 1, 2$) with $c_i(0) = c_i'(0) = 0$. Motivated by the literature on tournaments with heterogeneous agents, e.g. Lazear and Rosen (1981), we now allow individual minimum distances $\delta_i, i = 1, 2$. Agent $i$ receives the winner prize $w_H$ and the other agent, $j$, receives the loser prize $w_L$ if $x_i(e_i) > x_j(e_j) + \delta_i$. If $x_i(e_i) \leq x_j(e_j) + \delta_i$ for $i = 1$ and $i = 2$, a tie occurs and each
agent receives \((w_L + w_H)/2\). To rule out that both agents get \(w_H\), we assume that \(\delta_1 + \delta_2 \geq 0\). This class of contracts generalizes the class of contracts with handicaps by allowing the possibility of ties. Under individual contracts with \(\delta_1 + \delta_2 = 0\), ties do not occur.

A straightforward modification of the argument in Section 3 shows that the principal will always choose \(w_L = 0\) and that for given contracts \((w_H, w_L, \delta_i)\) with \(w_L = 0\), equilibrium efforts are given by the incentive constraint

\[
\frac{w_H}{2} \left[ g(h(e_2) - h(e_1) + \delta_1) + g(h(e_2) - h(e_1) - \delta_2) \right] = \frac{c'_1(e_1)}{h'(e_1)} = \frac{c'_2(e_2)}{h'(e_2)}.
\]

The principal chooses \(\delta_1, \delta_2\) and \(w_H\) to maximize \(h(e_1) + h(e_2) - w_H\) subject to this constraint. Let \(e^*_1\) and \(e^*_2\) denote the equilibrium efforts under the optimal contracts. To minimize the labor costs for these efforts the principal must choose distances \(\delta^*_i\) that maximize

\[
g(h(e^*_2) - h(e^*_1) + \delta_1) + g(h(e^*_2) - h(e^*_1) - \delta_2)
\]

subject to \(\delta_1 + \delta_2 \geq 0\). We will now use this characterization to compare in two settings the optimal distances when the cost functions differ to the optimal distance when the cost functions coincide.

Consider first the binary model of Section 4 and suppose that under identical cost functions, the unique optimal contract has a common minimum distance \(\delta^* > 0\). Then \(g(\delta^*) = g(-\delta^*) > g(x)\) for all \(x \in \mathbb{R} \setminus \{-\delta^*, \delta^*\}\), see the proof of Proposition 2. Hence, under individual cost functions \(c_i\), optimal individual \(\delta^*_i\) are given by \(\delta^*_1 = \delta^* + h(e^*_1) - h(e^*_2)\) and \(\delta^*_2 = \delta^* + h(e^*_2) - h(e^*_1)\). Several properties of the optimal contracts with individual \(\delta^*_i\) can be derived from this representation. (i) The contracts are again contracts with ties. Specifically, the probability of a tie under these contracts is the same as the probability of a tie under the optimal contract with a common distance \(\delta^*\) when the cost functions coincide:

\[
P(-\delta^*_2 \leq x_1(e^*_1) - x_2(e^*_2) \leq \delta^*_1) = P(|\Delta \eta + \sigma \Delta \epsilon| \leq \delta^*) > 0.
\]

(ii) If the cost functions do not differ much so that \(e^*_1\) and \(e^*_2\) are not far apart, then the
individual $\delta^*_i$ are close to $\delta^*$. These properties show the robustness of Proposition 2 to the assumption of homogeneous agents. (iii) If $c'_1(e) > c'_2(e)$ for all $e > 0$, that is, if agent 1 has higher marginal costs than agent 2, then $e^*_1 < e^*_2$, and it follows that $\delta^*_1 < \delta^* < \delta^*_2$. The agent having the cost disadvantage is encouraged by a smaller minimum distance by which he has to beat the other agent.

Consider next the setting of Proposition 7 (or the binary model with $P(\eta_1 \neq \eta_2) < \frac{2}{3}$) where under identical cost functions the standard contract without tie is optimal. Then $g(0) > g(x)$ for all $x \neq 0$, and it follows that $\delta^*_1 = h(e^*_1) - h(e^*_2)$ and $\delta^*_2 = h(e^*_2) - h(e^*_1)$. Thus, $\delta^*_1 + \delta^*_2 = 0$, so that under the optimal individual contracts, ties will not occur either. In the present case, these contracts are just the standard contracts with handicaps and without ties. Again, if $c'_1(e) > c'_2(e)$ for all $e > 0$, then $\delta^*_1 < 0 < \delta^*_2$.

To summarize, introducing the possibility of a tie and adding handicaps are two means to even out the competition between agents. Ties reduce ex-post unbalanced-ness, whereas handicaps can only reduce ex-ante heterogeneity. The present article focuses on the question when ties are useful, but the binary example has shown how both means can be combined.

\[ \square \] State-dependent total compensation. We have assumed that the outputs or performance measures $x_i(e_i)$ are unverifiable. As a consequence, the total compensation, that is, the sum of the prizes paid to the agents, must always be the same irrespective of whether an agent wins by the minimum distance $\delta$ or not. However, if the $x_i(e_i)$ are verifiable, the prizes paid in case of a tie may be chosen independently of the winner and the loser prize. In the following, we will discuss the impact of the additional contractual freedom on the principal’s choice between a contract with and one without ties.

If, under unverifiable outputs, the standard tournament contract can be improved by the possibility of ties, then, of course, at least the same improvement over the standard contract can be obtained by introducing ties when outputs are verifiable.
Thus, all our results where we showed that introducing possible ties is optimal under the assumption of unverifiable outputs continue to hold without this assumption.

If, under unverifiable outputs, the standard tournament contract cannot be improved by introducing ties, because ties would reduce incentives, it is still possible that the principal benefits from ties when outputs are verifiable. We now assume outputs are verifiable and address the question of when and why ties are useful even if they reduce incentives. Let again \( w_H \) and \( w_L \) denote the winner and the loser prize. Suppose now that in case of a tie, each agent receives \( w_T \), which may differ from \( \frac{(w_H + w_L)}{2} \). Given these prizes, the agents’ problem from Section 3 has to be modified as follows. Agent 1 maximizes

\[
EU_1(e_1) = w_L \cdot G(h(e_2) - h(e_1) - \delta) + w_H \cdot [1 - G(h(e_2) - h(e_1) + \delta)]
+ w_T \cdot [G(h(e_2) - h(e_1) + \delta) - G(h(e_2) - h(e_1) - \delta)] - c(e_1)
\]

and agent 2

\[
EU_2(e_2) = w_L \cdot [1 - G(h(e_2) - h(e_1) + \delta)] + w_H \cdot G(h(e_2) - h(e_1) - \delta)
+ w_T \cdot [G(h(e_2) - h(e_1) + \delta) - G(h(e_2) - h(e_1) - \delta)] - c(e_2).
\]

Assuming again that agents’ optimal efforts are described by the first-order conditions, the equilibrium solution \((e_1^*, e_2^*)\) is implicitly described by

\[
(w_T - w_L)g(\Delta h - \delta) + (w_H - w_T)g(\Delta h + \delta) = \frac{c'(e_1^*)}{h'(e_1^*)} \quad (10)
\]

and

\[
(w_T - w_L)g(\Delta h + \delta) + (w_H - w_T)g(\Delta h - \delta) = \frac{c'(e_2^*)}{h'(e_2^*)} \quad (11)
\]

with \( \Delta h := h(e_2^*) - h(e_1^*) \). The principal maximizes

\[
h(e_1^*) + h(e_2^*) - (w_H + w_L)(1 - [G(\Delta h + \delta) - G(\Delta h - \delta)])
- 2w_T[G(\Delta h + \delta) - G(\Delta h - \delta)]
\]

subject to the limited-liability constraint \( w_H, w_L, w_T \geq 0 \) and the incentive constraints (10) and (11).
We immediately obtain $w_L = 0$ as the optimal loser prize, but a complete characterization of the general solution would be quite intricate. We now concentrate on a typical case that leads to a clear rejection of ties when outputs are unverifiable, namely the case considered by Lazear and Rosen (1981): Suppose, as in Proposition 7 a), that the $\theta_i$ are i.i.d. so that $g$ is symmetric about zero, and that the homogeneous agents choose the same effort in equilibrium, i.e., $e_1^* = e_2^* = e^*$. Using the fact that $g(\delta) = g(-\delta)$ and $G(-\delta) = 1 - G(\delta)$, the principal’s problem reduces to

$$\max_{w_H, w_T, \delta \geq 0} 2h(e^*) - 2w_H [1 - G(\delta)] - 2w_T [2G(\delta) - 1] \text{ s.t. } w_H g(\delta) = \frac{c'(e^*)}{h'(e^*)}.$$  \hspace{1cm} (12)

As $w_T$ does not influence incentives and as choosing $w_T > 0$ leads to additional expected labor costs for the principal, $w_T = 0$ is optimal. Therefore, the problem further reduces to

$$\max_{\delta \geq 0} 2h(e^*) - 2w_H [1 - G(\delta)] - 2w_T [2G(\delta) - 1] \text{ s.t. } w_H g(\delta) = \frac{c'(e^*)}{h'(e^*)}.$$  \hspace{1cm} (12)

where $\lambda(\delta) = g(\delta)/[1 - G(\delta)]$ denotes the hazard rate of $\theta_1 - \theta_2$.\footnote{See Imhof and Kräkel (2014a) for a more detailed discussion.} This shows that the principal strictly benefits from choosing $\delta > 0$ if and only if $\lambda(\delta) > \lambda(0)$. For some distributions, e.g. for the normal distribution, this inequality is satisfied for every $\delta > 0$, so that every positive $\delta$ leads to an improvement. A contract without ties will only be optimal if the hazard rate attains its maximum over $[0, \infty)$ at 0.

The economic intuition for the choice of $\delta$ can best be seen from (12). On the one hand, the incentive constraint shows that equilibrium efforts increase in $g(\delta)$. As $g$ has a global maximum at 0, choosing $\delta > 0$ is detrimental from the incentive perspective. On the other hand, the principal’s objective function shows that the probability of paying out the winner prize, $2[1 - G(\delta)]$, decreases in $\delta$. Hence, from the labor cost perspective, choosing $\delta > 0$ is beneficial. If the second effect outweighs the first (e.g., in case of normally distributed $\theta_i$), the principal will introduce possible ties.

\textbf{Incentives versus promotion.} We have assumed that the principal chooses a contract to maximize his immediate profit and the object of the tournament has been
to create incentives. Now we consider a promotion tournament (e.g., Prendergast 1992, 1993), so that the principal is also interested in selecting a suitable agent. Suppose that a vacancy must be filled via promotion and that the $\eta_i$ denote the agents’ ex ante unknown effective abilities whose realizations are decisive for the vacant job. Then optimal promotion requires that the principal updates his expectations on $\eta_i$ against the background of realized performance and promotes the agent with the higher expected ability. Suppose that the principal has introduced a possible tie to improve incentives. If a tie occurs, the following conflict between the incentive goal and the promotion goal will arise. On the one hand, the principal might have sufficient commitment power and sticks to the specified tie-breaking rule – e.g., promoting at random. In this case, the incentive goal is satisfied but the promotion goal will be violated, if the promoted agent is not the one with the higher expected ability. If, on the other hand, the principal lacks sufficient commitment power, he will ex post always promote the agent with the higher expected ability. In that case, the promotion goal is satisfied but tournament incentives and the effect of ties are eliminated.

If, however, the vacancy need not be filled at any cost, the introduction of ties can serve both incentive and promotion purposes, because ties can reduce the risk of promoting an unsuitable agent. Suppose the object of the tournament is to identify a suitable agent. We again interpret the $\eta_i$ as ex ante unknown effective abilities and assume that their realizations are relevant for the vacant job, so that the principal is interested in promoting an agent with a high realization of $\eta_i$. We assume that at most one agent can be promoted. Receiving the winner prize $w_H$ is now identified with being promoted, so that agent $i$ will be promoted if $x_i(e_i) > x_j(e_j) + \delta, j \neq i$. In case of a tie, neither agent is promoted.\(^{26}\) We are interested in the case where this

\(^{26}\)This decision corresponds to what has been shown to be optimal when outputs are verifiable, where both agents receive $w_T = w_L = 0$ in case of tie, see the subsection “State-dependent total compensation.” The argument that led us to consider only contracts that satisfy the self-commitment property when outputs are unverifiable does not go through when the aim is to select an agent for promotion. Below we also consider the alternative rule where in case of a tie one agent is chosen at
last decision is costly to the principal compared to promoting a suitable agent, but is less costly than promoting an unsuitable agent.

We formulate the principal’s decision problem as a Bayesian decision problem (see also Meyer 1991, Prendergast 1992, Fairburn and Malcomson 2001) and examine how the minimum distance can strike the balance between the risk of promoting an unsuitable agent and the cost due to promoting no agent. We restrict the analysis to the binary model of Section 4 and we regard agent \(i\) as suitable if and only if \(\eta_i = a\). Aiming at promoting a suitable agent, the principal can incur two types of losses: \(\ell_1 > 0\) if he promotes an unsuitable agent, and \(\ell_0 > 0\) if he promotes neither agent. The probability of incurring loss \(\ell_0\) increases in \(\delta\), but the probability for loss \(\ell_1\) decreases. The principal incurs no loss if he promotes a suitable agent. The Bayes risk under the binary prior distribution on the agents’ suitabilities is

\[
r(\delta) = \ell_0 P(|\Delta x| \leq \delta) + \ell_1 [P(\Delta x > \delta, \eta_1 = 0) + P(\Delta x < -\delta, \eta_2 = 0)],
\]

where \(\Delta x = x_1(e_1) - x_2(e_2)\) and equilibrium efforts \(e_i\), computed as in the subsection above on state-dependent total compensation, are assumed to coincide. If the principal uses a contract without ties, then he will not incur cost \(\ell_0\) and the Bayes risk is

\[
r(0) = \ell_1 [P(\eta_1 = \eta_2 = 0) + P(\sigma \Delta \epsilon > a) P(\eta_1 \neq \eta_2)].
\]

Whether the risk can be reduced by choosing a positive minimum distance \(\delta\) depends on the ratio \(\ell_1/\ell_0\). The larger this ratio, the more the principal will benefit from reducing the risk of promoting an unsuitable agent by choosing \(\delta > 0\). In the appendix we determine explicitly the critical ratio \(\rho^* \geq 1\) so that the principal can benefit from choosing a contract with ties if and only if \(\ell_1/\ell_0 > \rho^*\). A simple sufficient condition for this inequality is that

\[
\frac{\ell_1}{\ell_0} > 2 \quad \text{and} \quad \frac{\ell_1}{\ell_0} > 1 + \frac{P(\eta_1 = \eta_2 = a)}{P(\eta_1 = \eta_2 = 0)}.
\]  

(13)

E.g., if \(\eta_1\) and \(\eta_2\) are independent, the bound on the right-hand side becomes \(1 + [P(\eta_1 = a)/P(\eta_1 = 0)]^2\) and so the principal can benefit from choosing \(\delta > 0\) if random for promotion.
\( P(\eta_1 = a) \leq \frac{1}{2} \) and \( \ell_1/\ell_0 > 2 \). In words, a principal will benefit from possible ties in promotion decisions if suitable agents are sufficiently scarce and the costs of promoting an unsuitable agent are sufficiently large compared to the costs of a vacancy.

It is natural to consider also contracts where, in case of a tie, the principal promotes one of the agents at random to avoid loss \( \ell_0 \). The Bayes risk for these contracts is

\[
\tilde{r}(\delta) = \ell_1 \left[ P(\Delta x > \delta, \eta_1 = 0) + P(\Delta x < -\delta, \eta_2 = 0) + \frac{1}{2} P(|\Delta x| \leq \delta, \eta_1 = 0) + \frac{1}{2} P(|\Delta x| \leq \delta, \eta_2 = 0) \right].
\]

A simple argument based on the unimodality of \( \Delta \epsilon \) shows that the risk function \( \tilde{r} \) is increasing. In agreement with Blackwell’s theorem,\(^{27}\) the randomization in case of a tie reduces the information obtained from observing the outputs \( x_i(e_i) \) and this disadvantage is not outweighed by avoiding loss \( \ell_0 \). Among these contracts, the one without ties, i.e. with \( \delta = 0 \), has the smallest risk. Moreover, if \( \delta^* \) minimizes \( r(\delta) \), then for every \( \delta \geq 0 \), \( r(\delta^*) \leq r(0) = \tilde{r}(0) \leq r(\delta) \). That is, the best contract without promotion in case of a tie cannot be improved by one of the contracts with random promotion in case of a tie.

\[\square\] **Performance evaluation by a supervisor.** Since the seminal article by Tirole (1986), there has been an extensive discussion of vertical collusions and side payments in a principal-supervisor-agent framework. In the Tirole model, in some states of the world, the supervisor has better information on the agent’s work environment than the principal. The supervisor can report his information to the principal and reported information is hard or verifiable, but the supervisor can be bribed by the agent to withhold information when observing an advantageous work environment. In this setting, the principal’s optimal contract has to satisfy additional incentive constraints that prevent profitable collusions between the supervisor and the agent. The additional constraints at least weakly harm the principal compared to a situation without collusion. The work subsequent to Tirole (1986) offers many variations of

\(^{27}\) See Blackwell (1951), Laffont (1989), and Laffont and Martimort (2002).
the collusion theme, including the possibility of multiple agents (Laffont 1988, 1990), multiple supervisors (Kofman and Lawarrée 1996) and soft instead of hard information (Faure-Grimaud et al. 2003).

If, in our setting, a supervisor instead of the principal has to evaluate the two agents, he can force them into a hidden game, similar to the situation considered by Laffont (1988, 1990). In the model of Laffont, the principal only obtains hard information about the sum of both agents’ outputs, but the supervisor additionally learns the agents’ individual performance. Let this assumption also hold for our tournament setting. Then the supervisor could demand side payments from the agents by credibly threaten to decrease the reported output of one agent by a certain amount and add this amount to the reported output of the other agent. Such extortion would result in an appropriation of the agents’ rents from participating in the tournament.\footnote{Recall that, due to limited liability, the agents’ participation constraints are non-binding under the optimal contract.}

The introduction of a minimum distance $\delta > 0$ may make a difference in case of hidden gaming, as such a distance tends to restrict the scope for collusion. Suppose both agents choose identical efforts in equilibrium, leading to similar outputs. In that situation, it would be easy for the supervisor to favor one of the agents and declare him the winner in a standard tournament. The introduction of a minimum distance, however, works against such favoritism by forcing the supervisor to significantly exaggerate the observed performance of his favorite. Let, for example, $x_1 (e_1)$ and $x_2 (e_2)$ take only non-negative values. Suppose that the realized values satisfy $x_1 (e_1) + x_2 (e_2) < \delta$, which is observable by the principal. Then the supervisor cannot credibly claim that one agent has sufficiently outperformed the other one to obtain the winner prize, which would be possible under the standard tournament contract. In other words, the introduction of a minimum distance restricts the set of implementable side contracts between the supervisor and the agents.
9 Related Literature

The present article is related to the literature on optimal contracts under moral hazard when agents are risk neutral and protected by limited liability. This literature builds on the seminal articles by Sappington (1983) and Innes (1990), who introduced limited liability of the agents as contractual friction and highlighted the corresponding efficiency losses in contracting. Kim (1997) and Demougin and Fluet (1998) show under which conditions the optimal contract boils down to a simple bonus contract. As Jewitt et al. (2008) point out, the optimality of the bonus contract need not hold when agents are risk averse. Poblete and Spulber (2012) introduce a critical ratio defined as the product of the hazard rate of the state variable and the marginal rate of technical substitution. They show how the optimal contract depends on the characteristics of the critical ratio. Bose et al. (2010, 2011) add risk aversion to the setting with limited liability and solve for the optimal linear contract. Che and Yoo (2001), Schmitz (2005a, 2013) and Ohlendorf and Schmitz (2012) analyze the specific problems of limited liability in a dynamic context where agents have to perform successive tasks. In contrast to all these articles, we have focused on multi-agent models with unverifiable performance measures.

Our article contributes to the ongoing debate on ties in tournaments. The idea of introducing a minimum distance between the best and the second best agents traces back to Nalebuff and Stiglitz (1983) who investigate under which conditions such gap is optimal given risk averse agents and competitive markets, which yield a zero-profit condition for firms in equilibrium.

Cohen and Sela (2007) show that, in their model, a possible tie clearly harms incentives. However, their assumptions differ in several respects from ours. First, they analyze an all-pay auction with discrete efforts and complete information. Second, the principal wants to maximize the expected total effort, whereas in our article the principal maximizes profits (i.e., output minus tournament prizes). Finally, contest prizes need not sum up to a certain collective amount of money, whereas in our
model the sum of the prizes is fixed to guarantee the self-commitment property of a tournament highlighted in the introduction.

Similar to our article, Eden (2007) analyzes a Lazear-Rosen type of tournament. She shows that introducing a possible tie will be optimal if the prizes are exogenous and if they need not satisfy the self-commitment property. However, if tournament prizes must always sum up to the same constant, the standard tournament contract without tie will be optimal. Eden assumes that idiosyncratic shocks to output are independent and symmetrically distributed about zero. For this special case, our model also leads to the optimality of the standard tournament contract.

Kono and Yagi (2008) consider a related tournament model without ties and speculate that introducing a possible tie may increase agents’ incentives. We show that this guess is correct under certain conditions and we obtain new insights into their approach to increasing incentives by reducing the accuracy of the performance measures. Imhof and Kräkel (2014a) compare individual incentive contracts and tournament contracts with ties under the assumption that performance signals are objective. Imhof and Kräkel (2014b) consider a tournament model with risk averse agents and show that ties improve the standard tournament contract only if agents are not protected by limited liability.

None of the articles on ties in tournaments considers the problem of ex-post unbalanced competition. Our article incorporates this natural problem into tournament theory and shows under what conditions the introduction of a tie can improve the incentive effects of a standard tournament.

10 Conclusion

We have analyzed a tournament model with two agents in which an agent’s performance measure depends on his effort, an unpredictable competitive advantage, and idiosyncratic noise. The performance measure is described by a continuous random variable, so that pure tournament competition would always lead to a clear-cut result.
– the best agent would get the winner prize, the second-best the loser prize, and ties would never occur. However, our results show that, in many situations, pure tournament competition can be strictly improved by introducing ties. For this purpose, the principal fixes a minimum distance by which the best agent must beat the second-best to obtain the winner prize. Otherwise neither agent will be declared winner and prizes are equally shared or distributed by tossing a coin. Endogenous ties are optimal whenever it is not unlikely that competition will be unbalanced ex post and exogenous noise does not suffice to counterbalance the uneven situation.

Ex-post unbalancedness arises, e.g., (i) when agents’ effective abilities for completing an innovative task turn out to differ, (ii) when agents face asymmetric market conditions after having started in a fair tournament, and (iii) when one agent benefits from an advantage due to an unconscious bias of the principal or a supervisor. Under the strong assumption that ex-post unbalancedness will not occur, ties, as well as imprecise measurements, reduce incentives. Otherwise, both ties and imprecise measurements can increase incentives and ties are always more effective. Ties ensure that being advantaged is not sufficient to obtain the winner prize and that being disadvantaged does not imply losing. Each agent still has to exert considerable effort to win or to avoid losing, respectively.

Appendix

The appendix contains proofs of the claims in the main text. Recall our assumptions that

(i) the noise terms $\epsilon_1, \epsilon_2$ are i.i.d. and $(\epsilon_1, \epsilon_2)$ is independent of $(\eta_1, \eta_2)$;

(ii) $\Delta \eta + \sigma \Delta \epsilon = \eta_1 + \sigma \epsilon_1 - \eta_2 - \sigma \epsilon_2$ has a continuous density $g$.

Both conditions are assumed throughout the appendix except in the following lemma. The lemma provides a simple condition which ensures that the second assumption holds and that an optimal minimum distance exists.
Lemma 1 Suppose independence condition (i) is satisfied and the noise terms $\epsilon_i$ have a square-integrable density. Then, for every $\sigma > 0$, $\Delta \eta + \sigma \Delta \epsilon$ has a continuous density and there exists an optimal minimum distance $\delta^* \in [0, \infty)$.

Proof. Denote the density of $\epsilon_i$ by $f$. Then the density of $\Delta \epsilon$ is given by the convolution $f_{\Delta \epsilon}(x) = \int f(y)f(y-x)\,dy$, see e.g. formula (20.38) in Billingsley (1995). As $f$ is square-integrable, $f_{\Delta \epsilon}$ is continuous and $\lim_{|x| \to \infty} f_{\Delta \epsilon}(x) = 0$, see Theorem 21.33 in Hewitt and Stromberg (1969). The density of $\Delta \eta + \sigma \Delta \epsilon$ is given by $g(y) = \sigma^{-1} \int f_{\Delta \epsilon}((y-x)/\sigma) \, d\mu(x)$, where $\mu$ is the distribution of $\Delta \eta$, see formula (20.37) in Billingsley (1995). It follows by bounded convergence that $g$ is continuous and $\lim_{|x| \to \infty} g(x) = 0$. Hence $g(\delta) + g(-\delta)$ attains its maximum in $[0, \infty)$ and every point $\delta^*$ where the maximum is attained is, by Proposition 1, an optimal minimum distance. 

Proof of Proposition 2. The proof uses the following auxiliary result for the normal density $\phi(x) = \exp(-x^2/2)/\sqrt{2\pi}$ evaluated at three equidistant points.

Lemma 2 Let $a > 0$ and

$$\psi(x) = \phi(x-a) + \phi(x) + \phi(x+a).$$

Then $\psi(x) < \psi(0)$ for all $x \neq 0$.

Proof. As $\phi$ is strictly decreasing on $[0, \infty)$, $\psi$ is strictly decreasing on $[a, \infty)$, so that $\psi(x) < \psi(a)$ for all $x > a$. If $x \in (a/2, a]$, then $a-x \in [0, a/2)$ and $0 \leq 2a-x < x+a$, so that

$$\psi(a-x) = \psi(x) + \phi(2a-x) - \phi(x+a) > \psi(x).$$

Hence $\psi(x) < \max_{y \in [a/2]} \psi(y)$ for all $x > a/2$. We will now show that $\psi$ is strictly decreasing on $[0, a/2]$. It will then follow that $\psi(0) > \psi(x)$ for all $x > 0$ and so, as $\psi(x) = \psi(-x)$, $\psi(0) > \psi(x)$ for all $x \neq 0$. We have

$$\psi'(x) = \phi(x) e^{-a^2/2} s(x),$$
where
\[ s(x) = -xe^{a^2/2} + 2a \sinh ax - 2x \cosh ax = \sum_{k=0}^{\infty} c_k x^{2k+1} \]
and
\[ c_0 = -e^{a^2/2} + 2(a^2 - 1), \quad c_k = \frac{a^{2k}}{(2k)!} \left( \frac{a^2}{2k+1} - 1 \right), \quad k \geq 1. \]

For every \( a \),
\[ c_0 < -\sum_{k=0}^{3} \frac{1}{k!} \left( \frac{a^2}{2} \right)^k + 2(a^2 - 1) \]
\[ = \frac{7\sqrt{7} - 19}{3} - \frac{(a^2 + 2 + 4\sqrt{7})(a^2 + 2 - 2\sqrt{7})^2}{48} < 0. \]

If \( a^2 < 3 \), then \( c_k < 0 \) for all \( k \geq 0 \), so that \( \psi'(x) < 0 \) for all \( x > 0 \).

Suppose next that \( 3 \leq a^2 \leq 4 \). Then \( c_1 \geq 0 \) and \( c_k \leq 0 \) for all \( k \geq 2 \). Hence for all \( x \in (0, a/2] \),
\[ \frac{s(x)}{x} \leq c_0 + c_1 \frac{a^2}{4} \leq -\sum_{k=0}^{6} \frac{a^{2k}}{2^{k+1}k!} - 2 + a^2 + \frac{a^4}{3} : w(a), \]
say. An application of Descartes’ rule of signs (see Chapter 1 of Part V in Pólya and Szegö, 1976) shows that \( w' \) has exactly one positive zero, and as \( w'(2) = 2/15 > 0 \) and \( \lim_{x \to \infty} w'(x) = -\infty \), \( w' \) must be strictly positive on \( (0, 2) \). Hence for all \( x \in (0, a/2] \),
\[ s(x)/x \leq w(a) \leq w(2) = -1/45 < 0. \]

Suppose finally that \( a^2 > 4 \). Then \( c_k > 0 \) if \( 1 \leq k < (a^2 - 1)/2 \) and \( c_k < 0 \) if \( k > (a^2 - 1)/2 \). By the version of Descartes’ rule of sign for series, \( s' \) has at most two positive zeros. As \( s'(0) = c_0 < 0 \), and
\[ s' \left( \frac{a}{2} \right) = \left( -2 + \frac{a^2}{2} \right) e^{a^2/2} + \left( \frac{3a^2}{2} - 1 \right) e^{-a^2/2} > 0, \]
\( s' \) has exactly one zero in \( (0, a/2] \) and \( s' \) changes sign from minus to plus at this zero.

Hence for all \( x \in (0, a/2] \),
\[ s(x) < \max \left\{ s(0), s \left( \frac{a}{2} \right) \right\} = \max \left\{ 0, -\frac{3}{2} ae^{-a^2/2} \right\} = 0. \]
The proof of Proposition 2 proceeds as follows. The density of \( \sigma \Delta \epsilon \) is given by 
\[
\phi(x/\sqrt{2\sigma})/\sqrt{2\sigma},
\]
and so the density of \( \Delta \eta + \sigma \Delta \epsilon \) is given by
\[
g(x) = \sum_{y \in \{-a,0,a\}} \frac{1}{\sqrt{2\sigma}} \phi\left(\frac{x-y}{\sqrt{2\sigma}}\right) P(\Delta \eta = y).
\]
By (5),
\[
P(\Delta \eta = a) = P(\eta_2 = 0) - P(\eta_1 = 0, \eta_2 = 0) = P(\eta_1 = 0) - P(\eta_1 = 0, \eta_2 = 0) = P(\Delta \eta = -a),
\]
so that 
\[
P(\Delta \eta = a) = P(\Delta \eta = -a) = \frac{1}{2} P(\eta_1 \neq \eta_2).
\]
It follows that
\[
g(x) = \frac{P(\eta_1 \neq \eta_2)}{2\sqrt{2\sigma}} \psi(x) + \frac{2 - 3P(\eta_1 \neq \eta_2)}{2\sqrt{2\sigma}} \phi\left(\frac{x}{\sqrt{2\sigma}}\right),
\]
where
\[
\psi(x) = \phi\left(\frac{x-a}{\sqrt{2\sigma}}\right) + \phi\left(\frac{x}{\sqrt{2\sigma}}\right) + \phi\left(\frac{x+a}{\sqrt{2\sigma}}\right).
\]
By Lemma 2, \( \psi(x) < \psi(0) \) for all \( x \neq 0 \). Thus, if \( P(\eta_1 \neq \eta_2) \leq \frac{2}{3} \), then \( g(x) < g(0) \) for all \( x \neq 0 \), and so the optimal minimum distance must be zero by Proposition 1. If \( a \leq \sqrt{2\sigma} \), then \( P(|\eta_1 - \eta_2| \leq \sqrt{2\sigma}) = 1 \), and the optimal minimum distance is zero by Example 1. Suppose next that \( P(\eta_1 \neq \eta_2) > \frac{2}{3} \) and that \( \sigma \) satisfies inequality (6). Then
\[
g(a) - g(0) > \frac{P(\eta_1 \neq \eta_2)}{2\sqrt{2\sigma}} \phi(0) - g(0)
\]
\[
= \frac{1}{2\sigma \sqrt{\pi}} \left[ P(\eta_1 \neq \eta_2) \left( \frac{3}{2} - e^{-a^2/(4\sigma^2)} \right) - 1 \right]
\]
\[
\geq 0.
\]
Hence \( g(a) + g(-a) = 2g(a) > 2g(0) \), so that a standard tournament contract without tie is not optimal. As \( \lim_{x \to \infty} g(x) + g(-x) = 0 \), there exists an optimal minimum distance, which must be positive. ■
Proof of Proposition 3. As we consider only contracts with distance \( \delta = 0 \), a level \( \sigma^* \in [\sigma_0, \infty) \) is optimal if and only if \( g_{\sigma^*}(0) \geq g_\sigma(0) \) for all \( \sigma \in [\sigma_0, \infty) \). Let

\[
H(u) = -\frac{P(\eta_1 = \eta_2)}{2} - \frac{P(\eta_1 \neq \eta_2)}{\sqrt{e}}ue^u
\]

for all \( u \in \mathbb{R} \). Then, by (14),

\[
\frac{\partial}{\partial \sigma} g_\sigma(0) = \frac{1}{\sigma^2 \sqrt{\pi}} H \left( \frac{1}{2} - \frac{\sigma^2}{4\sigma^2} \right).
\]

If \( P(\eta_1 \neq \eta_2) > 0 \), then \( H \) is strictly increasing on \((-\infty, -1]\) and strictly decreasing on \([-1, \infty)\). Moreover,

\[
\max_u H(u) = H(-1) = -\frac{1}{2} + P(\eta_1 \neq \eta_2) \left( \frac{1}{2} + e^{-3/2} \right).
\]

a) Suppose \( P(\eta_1 \neq \eta_2) \leq 1/(1 + 2e^{-3/2}) \). Then \( H(u) < 0 \) for all \( u \neq -1 \), and so \( (\partial/\partial \sigma)g_\sigma(0) < 0 \) on \((0, \infty)\) except possibly at \( \sigma = a/\sqrt{6} \). Hence \( g_{\sigma_0}(0) > g_{\sigma}(0) \) for all \( \sigma > \sigma_0 \).

b) Suppose \( 1/(1 + 2e^{-3/2}) < P(\eta_1 \neq \eta_2) < 1 \). Then \( H(-1) > 0 \) and

\[
\lim_{u \to -\infty} H(u) = H(0) = -\frac{P(\eta_1 = \eta_2)}{2} < 0.
\]

Thus, \( H \) has exactly two zeros, one in \((-\infty, -1]\) and one in \((-1, 0)\). Therefore, \( (\partial/\partial \sigma)g_\sigma(0) \) has exactly two positive zeros, say \( \sigma_1 < a/\sqrt{6} \) and \( \sigma_2 > a/\sqrt{6} \). Moreover, \( (\partial/\partial \sigma)g_\sigma(0) < 0 \) on \((0, \sigma_1) \cup (\sigma_2, \infty)\) and \( (\partial/\partial \sigma)g_\sigma(0) > 0 \) on \((\sigma_1, \sigma_2)\). Define \( \bar{\sigma} \) by (8). Using that \( W(z)e^{W(z)} = z \) and \(-1 < W(z) < 0 \) for \(-e^{-1} < z < 0 \), one may verify that \( (\partial/\partial \sigma)g_\sigma(0) \mid_{\sigma = \bar{\sigma}} = 0 \) and \( \bar{\sigma} > a/\sqrt{6} \), so that \( \bar{\sigma} = \sigma_2 \). As \( g_\sigma(0) \) is strictly increasing on \([\sigma_1, \bar{\sigma}]\), \( g_{\sigma_1}(0) < g_{\bar{\sigma}}(0) \). As \( P(\eta_1 = \eta_2) > 0 \), \( \lim_{\sigma \to 0+} g_\sigma(0) = \infty \), and it follows that there is indeed a unique \( \bar{\sigma} \in (0, \bar{\sigma}] \) such that \( g_{\bar{\sigma}}(0) = g_\sigma(0) \). Moreover, \( g_\sigma(0) < g_{\bar{\sigma}}(0) \) for all \( \sigma \in (\bar{\sigma}, \bar{\sigma}] \) and \( (\partial/\partial \sigma)g_\sigma(0) < 0 \) on \((0, \bar{\sigma}) \cup (\bar{\sigma}, \infty)\). Thus, if \( \sigma_0 \in (0, \bar{\sigma}] \cup [\bar{\sigma}, \infty) \), then \( g_{\sigma_0}(0) \geq g_{\bar{\sigma}}(0) \) for all \( \sigma \geq \sigma_0 \). If \( \sigma_0 \in (\bar{\sigma}, \infty) \), then \( g_{\bar{\sigma}}(0) > g_{\sigma}(0) \) for all \( \sigma \in [\sigma_0, \infty) \setminus \{\bar{\sigma}\} \).

Suppose finally \( P(\eta_1 \neq \eta_2) = 1 \). Let \( \sigma = 0 \) and define \( \bar{\sigma} \) by (8), that is, \( \bar{\sigma} = a/\sqrt{2} \). Then \( (\partial/\partial \sigma)g_\sigma(0) > 0 \) on \((\bar{\sigma}, \bar{\sigma})\) and \( (\partial/\partial \sigma)g_\sigma(0) < 0 \) on \((\bar{\sigma}, \infty)\). Now the assertion follows as before. \( \square \)
Proof of Proposition 4. We prove the result under the weaker assumption that the noise terms $\epsilon_i$ have a non-degenerate stable distribution, see e.g. Section VI.1 in Feller (1971). This assumption implies that $\Delta \epsilon$ has a non-degenerate stable distribution as well. Therefore, the characteristic function of $\Delta \epsilon$ is integrable, so that $\Delta \epsilon$ has a continuous density $f_{\Delta \epsilon}$ and, by the Riemann-Lebesgue lemma, $\lim_{|x| \to \infty} f_{\Delta \epsilon}(x) = 0$, see Section XVII.6, Theorem 3 in Section XV.3, and Lemma 3 in Section XV.4 in Feller (1971). Let $\bar{g}_\sigma(x) = g_\sigma(x) + g_\sigma(-x)$ for all $\sigma > 0$ and $x \in \mathbb{R}$. It follows as in the proof of Lemma 1 that for each fixed $\sigma > 0$, $\lim_{|x| \to \infty} \bar{g}_\sigma(x) = 0$ and that $\bar{g}_\sigma(x)$ attains its maximum $M(\sigma)$ at some point $\delta^*_\sigma \geq 0$. In view of the representation of the principal’s maximum profit for each level $\sigma$ given in (4), we have to show that $M(\sigma)$ is strictly decreasing.

Fix $0 < \sigma_1 < \sigma_2$. Let $\Delta \tilde{\epsilon}$ be a random variable that has the same distribution as $\Delta \epsilon$ and assume that $\Delta \eta$, $\Delta \epsilon$ and $\Delta \tilde{\epsilon}$ are independent. As $\Delta \epsilon$ is stable and symmetric, there is a constant $\tilde{\sigma} > 0$ such that $\sigma_2 \Delta \epsilon$ has the same distribution as $\sigma_1 \Delta \epsilon + \tilde{\sigma} \Delta \tilde{\epsilon}$, see Theorem 3 in Section VI.1 in Feller (1971). Hence, $\Delta \eta + \sigma_2 \Delta \epsilon$ has the same distribution as $\Delta \eta + \sigma_1 \Delta \epsilon + \tilde{\sigma} \Delta \tilde{\epsilon}$. The density of $\tilde{\sigma} \Delta \tilde{\epsilon}$ is given by $f_{\Delta \epsilon}(y/\tilde{\sigma})/\tilde{\sigma}$, and so

$$g_{\sigma_2}(x) = \frac{1}{\sigma} \int g_{\sigma_1}(x - y) f_{\Delta \epsilon}\left(\frac{y}{\tilde{\sigma}}\right) \, dy.$$ 

As $f_{\Delta \epsilon}$ is an even function, it follows that

$$\bar{g}_{\sigma_2}(x) = \frac{1}{\sigma} \int \bar{g}_{\sigma_1}(x - y) f_{\Delta \epsilon}\left(\frac{y}{\tilde{\sigma}}\right) \, dy.$$ 

Therefore,

$$M(\sigma_1) - M(\sigma_2) = \frac{1}{\tilde{\sigma}} \int \{M(\sigma_1) - \bar{g}_{\sigma_1}(\delta^*_\sigma - y)\} f_{\Delta \epsilon}\left(\frac{y}{\tilde{\sigma}}\right) \, dy.$$ 

The term in braces is non-negative for all $y \in \mathbb{R}$ and positive if $|y|$ is sufficiently large. As the support of $f_{\Delta \epsilon}$ is unbounded (see Section VI.3 in Feller, 1971), it follows that $M(\sigma_1) > M(\sigma_2)$. ■
Proof of Proposition 5. Denote the continuous density of $|\Delta \eta + \sigma \Delta \epsilon|$ by $\overline{g}_\sigma$. As $0$ does not belong to the support of $\Delta \eta$, there exists $c_1 > 0$ such that $P(c_1 < |\Delta \eta|) = 1$. Let $c_2 > c_1$ be such that $P(c_1 < |\Delta \eta| < c_2) > 0$. As $\sigma \downarrow 0$, $|\Delta \eta + \sigma \Delta \epsilon|$ converges in law to $|\Delta \eta|$, and so
\[
\liminf_{\sigma \to 0} \int_{c_1}^{c_2} \overline{g}_\sigma(x) \, dx = \liminf_{\sigma \to 0} \frac{P(c_1 < |\Delta \eta| < c_2)}{P(c_1 < |\Delta \eta| < c_2)} \geq P(c_1 < |\Delta \eta| < c_2).
\]
It follows that for $\sigma$ sufficiently small there exists $y_\sigma \in (c_1, c_2)$ such that $\overline{g}_\sigma(y_\sigma) > P(c_1 < |\Delta \eta| < c_2)/(2(c_2 - c_1))$. To show that for small $\sigma$, minimum distance $y_\sigma$ is strictly better than minimum distance $0$, note first that
\[
\overline{g}_\sigma(0) = \frac{2}{\sigma} \int f_{\Delta \epsilon} \left( -\frac{x}{\sigma} \right) \, d\mu(x) \leq \frac{2}{\sigma} \sup_{|x| \geq c_1} f_{\Delta \epsilon} \left( -\frac{x}{\sigma} \right),
\]
where $f_{\Delta \epsilon}(x) = \int f(y) f(y-x) \, dy$ is the density of $\Delta \epsilon$ and $\mu$ is the distribution of $\Delta \eta$. Set $\gamma := c_1 P(c_1 < |\Delta \eta| < c_2)/(16(c_2 - c_1))$. By (9), there exists $K \in (0, \infty)$ such that $f(y) \leq \gamma/|y|$ for all $|y| \geq K$. Thus, if $x \geq 2K$, then
\[
f_{\Delta \epsilon}(x) = \int_{-\infty}^{\infty} f \left( y + \frac{x}{2} \right) f \left( y - \frac{x}{2} \right) \, dy \\
\leq \int_{-\infty}^{0} f \left( y + \frac{x}{2} \right) \frac{\gamma}{|y - \frac{x}{2}|} \, dy + \int_{0}^{\infty} f \left( y + \frac{x}{2} \right) \frac{\gamma}{y + \frac{x}{2}} \, dy \\
\leq \frac{2\gamma}{x} \int_{-\infty}^{0} f \left( y + \frac{x}{2} \right) \, dy + \frac{2\gamma}{x} \int_{0}^{\infty} f \left( y - \frac{x}{2} \right) \, dy \\
\leq \frac{4\gamma}{x}.
\]
Similarly, $f_{\Delta \epsilon}(x) \leq 4\gamma/|x|$ for $x \leq -2K$. Therefore, if $\sigma \leq c_1/(2K)$, then
\[
\overline{g}_\sigma(0) \leq \frac{2}{\sigma} \sup_{|x| \geq c_1} f_{\Delta \epsilon} \left( -\frac{x}{\sigma} \right) \leq \frac{2}{\sigma} \sup_{|x| \geq c_1} \frac{4\gamma}{|x|/\sigma} = \frac{8\gamma}{c_1} = \frac{P(c_1 < |\Delta \eta| < c_2)}{2(c_2 - c_1)}.
\]
Hence, for $\sigma$ sufficiently small, $\overline{g}_\sigma(y_\sigma) > \overline{g}_\sigma(0)$, so that in view of Proposition 1, minimum distance 0 is not optimal. \qed

Proof of Proposition 6. Denote the density of $\Delta \epsilon$ by $f_{\Delta \epsilon}$, the density of $|\Delta \eta + \sigma \Delta \epsilon|$ by $\overline{g}_\sigma$ and the distribution of $\Delta \eta$ by $\mu$. Let $\beta \sigma < \alpha$. If $|x| \geq \alpha$, then the convexity
assumption on $f_{\Delta}$ implies that $f_{\Delta}(y-x)/\sigma) + f_{\Delta}(-y+x)/\sigma)$ is strictly increasing in $y$ for $y \in [0, \alpha - \beta \sigma]$. Thus, if $0 \leq y_1 < y_2 \leq \alpha - \beta \sigma$, then

$$\|\sigma(y_1) = \frac{1}{\sigma} \int f_{\Delta}(y_1-x)/\sigma) + f_{\Delta}(-y_1-x)/\sigma) \, d\mu(x)$$

$$< \frac{1}{\sigma} \int f_{\Delta}(y_2-x)/\sigma) + f_{\Delta}(-y_2-x)/\sigma) \, d\mu(x)$$

$$= \|\sigma(y_2).$$

That is, $\|\sigma$ is strictly increasing on $[0, \alpha - \beta \sigma]$, and it now follows from Proposition 1 that every optimal minimum distance must belong to $[\alpha - \beta \sigma, \infty)$. ■

**Proof of Proposition 7.**

a) Suppose $\theta_1$ and $\theta_2$ are i.i.d. Then the density $g$ of $\Delta \theta := \theta_1 - \theta_2$ is an even function, so that in view of Proposition 1, a minimum distance is optimal if and only if it maximizes $g$. Let $\psi(x) = Ee^{ix\Delta \theta}$ be the characteristic function of $\Delta \theta$. For all $x \in \mathbb{R}$, $\psi(x) = E(e^{ix\theta_1})E(e^{-ix\theta_1}) = |Ee^{ix\theta_1}|^2 \geq 0$. Applying the pointwise summability theorem with Abel’s kernel (Theorem 21.43 and Example 21.45(a) in Hewitt and Stromberg, 1969), we obtain that

$$\lim_{n \to \infty} \frac{1}{2\pi} \int \psi(x)e^{-|x|/n} \, dx = g(0) < \infty,$$

where we used that $g$ is continuous at 0. As $\psi$ is non-negative, it follows by monotone convergence that the limit is $\frac{1}{2\pi} \int \psi(x) \, dx$. Thus, $\psi$ is integrable. The Fourier inversion theorem (Theorem 21.49 in Hewitt and Stromberg, 1969) is therefore applicable and so, as $g$ is continuous,

$$g(x) = \frac{1}{2\pi} \int e^{-itx} \psi(t) \, dt = \frac{1}{2\pi} \int \cos(tx) \psi(t) \, dt$$

for all $x \in \mathbb{R}$. Hence, for every $x$, $g(x) \leq \frac{1}{2\pi} \int |\cos(tx)|\psi(t) \, dt \leq \frac{1}{2\pi} \int \psi(t) \, dt = g(0).$

That is, 0 is an optimal minimum distance. One can even show uniqueness: Suppose $x$ is such that $g(x) = g(0)$. Then $|\cos(tx)|\psi(t) = \psi(t)$ for all $t \in \mathbb{R}$. As $\psi(t) > 0$ in a neighborhood of 0, it follows that $x = 0$. Thus, 0 is the unique optimal minimum distance.
b) and c) Under each of the conditions b) and c), \( g \) is also symmetric. We will show that \( g \) is unimodal. It follows that \( g \) attains its maximum at 0, so that a zero minimum distance is optimal.

According to a result due to Wintner (Theorem 1.6 in Dharmadhikari and Joag-dev, 1988), the convolution of two symmetric unimodal densities is unimodal. Thus under condition b), \( g \) is indeed unimodal. If \( \theta_1 \) and \( \theta_2 \) have a log-concave joint density, then any non-degenerate linear combination of \( \theta_1 \) and \( \theta_2 \), in particular \( \theta_1 - \theta_2 \), has a log-concave density, see Section 2.3 in Dharmadhikari and Joag-dev (1988). This shows that \( g \) is unimodal under c) as well.

**Proof of Proposition 8.** Let \( \mu \) denote the distribution of \( \Delta \eta \). Let \( f_{\Delta \epsilon} \) denote the density of \( \Delta \epsilon \); that is, \( f_{\Delta \epsilon}(y) = \int f(x)f(x+y)\,dx \). Then the density of \( \Delta \eta + \sigma \Delta \epsilon_1 \) is given by

\[
g_\sigma(y) = \frac{1}{\sigma} \int f_{\Delta \epsilon} \left( \frac{y-x}{\sigma} \right) \,d\mu(x).
\]

Let

\[
H_\sigma(y) := \sigma[g_\sigma(\sigma y) + g_\sigma(-\sigma y)].
\]

In view of Proposition 1, we have to show that \( H_\sigma(0) > H_\sigma(y) \) for all \( y \neq 0 \) when \( \sigma \) is large enough.

Using that \( f \) is integrable and that \( f' \) is bounded, one may verify by an application of Lebesgue’s dominated convergence theorem that for every \( y \in \mathbb{R} \),

\[
f'_{\Delta \epsilon}(y) = \int f(x)f'(x+y)\,dx = \int f(x-y)f'(x)\,dx.
\]

In particular, \( f'_{\Delta \epsilon} \) is bounded. Using now that \( f' \) is integrable, one may show similarly that for every \( y \in \mathbb{R} \),

\[
f''_{\Delta \epsilon}(y) = -\int f'(x-y)f''(x)\,dx.
\]

It follows that \( f''_{\Delta \epsilon} \) is bounded and continuous, see Theorem 21.33 in Hewitt and Stromberg (1969). We have \( f''_{\Delta \epsilon}(0) = -\int [f'(x)]^2\,dx \). If \( \int [f'(x)]^2\,dx = 0 \), then \( f' = 0 \) almost everywhere, and as \( f \) is absolutely continuous, it would follow that \( f \) is
constant. This is impossible because $f$ is a probability density. Hence $f''_{\Delta \epsilon}(0) < 0$. 

Thus there exists $x_0 > 0$ such that $f''_{\Delta \epsilon}(x) \leq f''_{\Delta \epsilon}(0)/2$ for all $x \in [-x_0, x_0]$. If $|y| \leq x_0/2$, then

$$
H''_{\sigma}(y) = \left\{ \int_{[-x_0 \sigma/2, x_0 \sigma/2]} + \int_{\mathbb{R}\setminus[-x_0 \sigma/2, x_0 \sigma/2]} \right\} f''_{\Delta \epsilon}\left(y - \frac{x}{\sigma}\right) + f''_{\Delta \epsilon}\left(-y - \frac{x}{\sigma}\right) \, d\mu(x)
$$

$$
\leq \mu\left(\left[-\frac{x_0 \sigma}{2}, \frac{x_0 \sigma}{2}\right]\right) f''_{\Delta \epsilon}(0) + 2 \sup_{x \in \mathbb{R}} f''_{\Delta \epsilon}(x),
$$

where we used that $f'_{\Delta \epsilon}$ and $f''_{\Delta \epsilon}$ are bounded. It follows that for $\sigma$ sufficiently large, $H_{\sigma}$ is strictly concave on $[-x_0/2, x_0/2]$, so that $H_{\sigma}(0) > H_{\sigma}(y)$ for all $y \in [-x_0/2, x_0/2] \setminus \{0\}$.

Let

$$M := \sup\{f_{\Delta \epsilon}(y - x) + f_{\Delta \epsilon}(-y - x) : x \in \mathbb{R}, y \in \mathbb{R} \setminus (-x_0/2, x_0/2)\}.$$

An application of the Cauchy-Schwarz inequality shows that $f_{\Delta \epsilon}(y) < f_{\Delta \epsilon}(0)$ for all $y \neq 0$. As $f$ is integrable and $f'$ is bounded, $f$ is bounded and, therefore, square-integrable. Consequently, $f_{\Delta \epsilon}$ is continuous and $\lim_{y \to \pm\infty} f_{\Delta \epsilon}(y) = \lim_{y \to \pm\infty} f_{\Delta \epsilon}(y) = 0$, see Theorem 21.33 in Hewitt and Stromberg (1969). It follows that $M < 2f_{\Delta \epsilon}(0)$. For all $y$ with $|y| > x_0/2$ and all $\sigma > 0$,

$$H_{\sigma}(y) = \int f_{\Delta \epsilon}\left(y - \frac{x}{\sigma}\right) + f_{\Delta \epsilon}\left(-y - \frac{x}{\sigma}\right) \, d\mu(x) \leq M.$$

On the other hand, by dominated convergence, $H_{\sigma}(0) \to 2f_{\Delta \epsilon}(0)$ as $\sigma \to \infty$, so that $H_{\sigma}(0) > M$ for $\sigma$ sufficiently large. \qed

**Proof of Example 1.** The proof rests on the following inequality for the normal density

$$\phi(x) = \exp(-x^2/2)/\sqrt{2\pi}. \quad \text{The inequality is a consequence of the known fact that the equally weighted mixture of two normals with a common variance is unimodal if the means of the components are at most two standard deviations apart; see, e.g., Lindsay (1995). We include a short proof for the sake of completeness.}

**Lemma 3** If $a \in [-1, 1]$, then

$$\phi(x + a) + \phi(x - a) < 2\phi(a) \quad \text{for all } x \neq 0.$$
Proof. Fix $a \in [-1, 1]$ and set $\psi(x) = \phi(x + a) + \phi(x - a)$. Then

$$
\psi'(x) = \frac{1}{\sqrt{2\pi}} e^{-(x^2 + a^2)/2} \psi_1(x),
$$

where $\psi_1(x) = (a - x)e^{ax} - (a + x)e^{-ax}$. As

$$
\psi'_1(x) = (a^2 - 1) (e^{ax} + e^{-ax}) - ax (e^{ax} - e^{-ax}) < 0
$$

for all $x \neq 0$ and $\psi_1(0) = 0$, it follows that $\psi'(x) > 0$ for $x < 0$ and $\psi'(x) < 0$ for $x > 0$. Hence $\psi(x) < \psi(0) = 2\phi(a)$ for all $x \neq 0$. □

Example 1 can now be proved as follows. Let $\mu$ denote the distribution of $\Delta \eta$. The density of $\sigma \Delta \epsilon$ is $\phi(x/(\sqrt{2}\sigma))/(\sqrt{2}\sigma)$, and so the density of $\Delta \eta + \sigma \Delta \epsilon$ is given by

$$
g(x) = \frac{1}{\sqrt{2\sigma}} \int \phi \left( \frac{x - y}{\sqrt{2\sigma}} \right) d\mu(y).
$$

Let $\overline{g}(x) = g(x) + g(-x)$ for all $x \geq 0$. If $|y| \leq \sqrt{2}\sigma$, then, by Lemma 3,

$$
\phi \left( \frac{x - y}{\sqrt{2\sigma}} \right) + \phi \left( \frac{-x - y}{\sqrt{2\sigma}} \right) < 2\phi \left( \frac{y}{\sqrt{2\sigma}} \right)
$$

for all $x \in \mathbb{R} \setminus \{0\}$. Hence, if $|\Delta \eta| \leq \sqrt{2}\sigma$ almost surely, then for all $x > 0,

$$
\overline{g}(x) = \frac{1}{\sqrt{2\sigma}} \int_{-\sqrt{2\sigma}, \sqrt{2\sigma}} \phi \left( \frac{x - y}{\sqrt{2\sigma}} \right) + \phi \left( \frac{-x - y}{\sqrt{2\sigma}} \right) d\mu(y) < \overline{g}(0),
$$

so that, by Proposition 1, the optimal minimum distance is zero. □

Proof of the claims in the subsection “Incentives versus promotion”. a) We first determine the critical ratio $\rho^*$. Let $p_0(\delta)$ denote the probability that a tie occurs, and let $p_1(\delta)$ denote the probability that an unsuitable agent is promoted. The Bayes risk $r(\delta)$ can then be written as

$$
r(\delta) = \ell_0 p_0(\delta) + \ell_1 p_1(\delta).
$$

As equilibrium efforts coincide, $\Delta x = \Delta \eta + \sigma \Delta \epsilon$, and it follows that

$$
p_0(\delta) = P(|\Delta x| \leq \delta)
$$

$$
= P(\sigma|\Delta \epsilon| \leq \delta)P(\eta_1 = \eta_2) + P(\sigma - \delta \leq \sigma \Delta \epsilon \leq \sigma + \delta)P(\eta_1 \neq \eta_2)
$$

$$
= [2F(\delta) - 1]P(\eta_1 = \eta_2) + [F(\sigma + \delta) - F(\sigma - \delta)]P(\eta_1 \neq \eta_2),
$$

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where $F$ is the distribution function of $\sigma \Delta \epsilon$. Similarly,

$$p_1(\delta) = P(\Delta x > \delta, \eta_1 = 0) + P(\Delta x < -\delta, \eta_2 = 0)$$

$$= P(|\Delta x| > \delta, \eta_1 = \eta_2 = 0) + P(\Delta x > \delta, \eta_1 = 0, \eta_2 = a)$$

$$+ P(\Delta x < -\delta, \eta_1 = a, \eta_2 = 0)$$

$$= P(\sigma |\Delta \epsilon| > \delta) P(\eta_1 = \eta_2 = 0) + P(\sigma \Delta \epsilon > a + \delta) P(\eta_1 \neq \eta_2)$$

$$= 2[1 - F(\delta)] P(\eta_1 = \eta_2 = 0) + [1 - F(a + \delta)] P(\eta_1 \neq \eta_2).$$

Obviously, $p_0(0) = 0$, and for every $\delta > 0$, $p_1(0) > p_1(\delta)$. Hence, for $\delta > 0$,

$$r(\delta) < r(0) \iff \frac{\ell_1}{\ell_0} > \frac{p_0(\delta)}{p_1(0) - p_1(\delta)}$$

Define

$$\rho^* = \inf_{\delta > 0} \frac{p_0(\delta)}{p_1(0) - p_1(\delta)}.$$

It follows that if $\ell_1/\ell_0 > \rho^*$, then there exists $\delta > 0$ with $r(\delta) < r(0)$, and if $\ell_1/\ell_0 \leq \rho^*$, then $r(\delta) \geq r(0)$ for all $\delta \geq 0$.

To see that $\rho^* \geq 1$ observe that $1 - p_0(\delta) - p_1(\delta)$, the probability that a suitable agent is promoted, is decreasing in $\delta$. Thus, for every $\delta > 0$, $p_0(\delta) + p_1(\delta) \geq p_1(0)$, and so $\rho^* \geq 1$.

b) We will next show that condition (13) implies that there exists $\delta > 0$ with $r(\delta) < r(0)$. We have

$$\rho^* \leq \lim_{\delta \to 0^+} \frac{p_0(\delta)}{p_1(0) - p_1(\delta)} = \frac{p_0'(0)}{p_1'(0)}$$

$$= \frac{F'(0)P(\eta_1 = \eta_2) + F'(a)P(\eta_1 \neq \eta_2)}{F'(0)P(\eta_1 = \eta_2) + \frac{1}{2} F'(a)P(\eta_1 \neq \eta_2)} \leq \max \left\{ \frac{P(\eta_1 = \eta_2)}{P(\eta_1 = \eta_2 = 0)}, 2 \right\},$$

where we used that $(A + B)/(C + D) \leq \max\{A/C, B/D\}$ for all $A, B, C, D > 0$. Hence, if (13) holds, then $\ell_1/\ell_0 > \rho^*$, and the claim follows from a).

c) To show that the Bayes risk $\tilde{r}(\delta)$ is increasing write

$$\tilde{r}(\delta) = \ell_1 [P(\eta_1 = \eta_2 = 0) + 2H(\delta)P(\eta_1 = a, \eta_2 = 0)],$$
where \( H(\delta) = \frac{1}{2} P(|a + \sigma \Delta \epsilon| \leq \delta) + P(a + \sigma \Delta \epsilon < -\delta) \) is the conditional probability of promoting agent 2 given that \( \eta_1 = a \) and \( \eta_2 = 0 \). We have

\[
2H(\delta) = P(a + \sigma \Delta \epsilon \leq \delta) + P(a + \sigma \Delta \epsilon < -\delta) = 1 - P(\sigma \Delta \epsilon \in [\delta - a, \delta + a]).
\]

As the distribution of \( \sigma \Delta \epsilon \) is symmetric and unimodal, \( P(\sigma \Delta \epsilon \in [\delta - a, \delta + a]) \) is a decreasing function of \( \delta \). It follows that \( H \) and \( \tilde{r} \) are increasing. ■

References


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