Optimal Tournament Contracts for Heterogeneous Workers∗

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Abstract

We consider a rank-order tournament between workers with different abilities. Under Malcomson’s self-commitment property, the employer optimally uses individual tournament prizes to extract rents from the workers. Without the self-commitment property, individual prizes are used to make the asymmetric competition less uneven by discriminating against the more able worker (i.e., he is offered a lower winner prize than the less able worker). In this spirit, individual prizes serve as a substitute for handicaps.

Key Words: heterogeneous workers, limited liability, rank-order tournaments, self commitment.

JEL Classification: D86, J33, M52.

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1 Introduction

Rank-order tournaments where workers compete for given prizes or the distribution of a fixed amount of bonuses are frequently used in practice. For example, managers face relative compensation schemes (Eriksson 1999), workers compete for job-promotion in corporate hierarchies (Baker et al. 1994), salesmen are compensated according to relative performance (Murphy et al. 2004), and workers compete for higher shares in bonus-pool arrangements (Rajan and Reichelstein 2006). Basically, rank-order tournaments always occur when relative worker performance is linked to monetary consequences. Thus, forced-ranking systems also belong to the class of tournament compensation schemes. Here, supervisors rate their subordinates according to relative performance given a fixed distribution of different grades that can be assigned to the workers (Boyle 2001).

In the literature, two major advantages of rank-order tournaments have been highlighted: (1) tournaments are implementable even in situations where performance information is only ordinal (Lazear and Rosen 1981). In many situations, the construction of reliable, cardinal performance measures is very costly. Therefore, the employer benefits from relying on ordinal information. (2) A tournament incentive scheme even works if the performance signal is not verifiable to a third party (Malcomson 1984). This self-commitment property of tournaments pointed out by Malcomson is based on the fact that tournament prizes are fixed in advance and that payment of these prizes is verifiable. Since, therefore, the employer must pay the high winner prize to one of the workers, he cannot save labor costs by misrepresenting the performance signal. In practice, we often have ordinal, unverifiable performance signals, in particular in connection with the performance evaluation of non-production or management tasks. From a theoretical point of view, ordinal and unverifiable performance signals render the use of explicit incentive schemes such as bonus pay or piece rates impossible.

The tournament literature shows that although a rank-order tournament is feasible under situations (1) and (2) mentioned before, the outcome will
be inefficient if workers are heterogeneous – even when the individual types of the workers are common knowledge: the efficient solution would only be implemented if the employer could use optimal handicaps to make the competition even.\(^1\) However, under ordinal information this is impossible as the employer cannot apply handicaps. In our paper, we consider such situation with an ordinal and unverifiable performance signal and workers that are characterized by observable heterogeneity. We show that, nevertheless, the employer may be able to implement efficient incentives without violating the important self-commitment property of Malcomson, mentioned above.

Our analysis of the optimal design of tournaments between two heterogeneous workers includes two steps. In the first part of the paper, we follow the previous tournament literature by analyzing a setting where the employer chooses uniform winner and loser prizes for the workers (i.e., winner and loser prizes do not depend on the identity of the respective winner or loser). Here, Malcomson’s self-commitment property is always satisfied. We show that even under unlimited liability a tournament with uniform prizes does not lead to first-best efforts. This implementation would be too costly for the employer as the more able worker still earns a positive rent. If workers are protected by limited liability, the employer may benefit from implementing more than first best efforts. He can use the worker competition to elicit high effort levels and, at the same time, to decrease the workers’ rents.

The standard case of uniform prizes identifies a fundamental dilemma of tournament theory: on the one hand, a tournament with uniform prizes exhibits Malcomson’s self-commitment property. On the other hand, it does not lead to first-best efforts so that there is an efficiency loss. In the second part of the paper, we show that tournaments with individual prizes that differ among workers (i.e., if worker A is declared tournament winner he may receive another prize than worker B in case of winning) can solve this dilemma. In particular, our results show that under unlimited liability the employer sets individual tournament prizes that (a) satisfy Malcomson’s self-commitment

constraint and (b) implement efficient effort levels. The employer benefits from individual prizes because they can be used to extract rents from the workers. Under limited liability, without self-commitment constraint individual prizes serve as a substitute for handicaps to adjust incentives so that the heterogeneous competition becomes less uneven.

The aim of our paper is twofold. First, we want to add to the theory of rank-order tournaments, as outlined before. Second, our theoretic analysis can be used to explain the empirical finding of considerable variation in pay on each hierarchy level in real firms. This stylized fact seems to be puzzling in the light of tournament theory since according to standard models on job-promotion tournaments wages must be attached to jobs and, thus, to hierarchy levels in order to generate incentives. However, the finding of pay variation on a single hierarchy level is quite in line with our result on individual tournament prizes dominating uniform prizes.

Individual prizes can also be found in many sports tournaments like in athletics, where the best athletes are typically paid an appearance fee in addition to a possible bonus for winning the competition. The perhaps most famous distance runner Haile Gebrselassie, for instance, receives about $250,000 in appearance money for marathons. The less famous runners, instead, do not receive appearance fees. As an appearance fee is the same as a loser prize in tournaments, the organizers of athletic events pay individual tournament prizes; high prizes for the famous athletes and lower prizes for the less famous ones. Individual prizes ensure the participation of the best athletes, while no rents are paid to the other ones. This is just one of the upsides to individual prizes that we emphasize in this paper.

Our paper is related to those few tournament models that also address the problem of heterogeneous workers. The seminal paper by Lazear and Rosen (1981) was the first one that points to the inefficient outcome of tournaments between heterogeneous workers. O'Keefe et al. (1984) investigate a

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3See the article "It's Only Half a Marathon, but Race is Full of the Best" in the New York Times on July 26, 2007.
tournament between heterogeneous workers under asymmetric information. There are two types of workers who have either high or low ability. Whereas each worker knows his individual type the employer lacks this information. O’Keeffe et al. show that optimal combinations of tournament prizes and monitoring precision can be used to restore incentives and make the workers self-sort into their respective type of tournament. Closely related to our paper is the work by Bhattacharya and Guasch (1988) who also analyze a situation with heterogeneous workers and efficient effort choices. They consider a setting where workers privately know their abilities and show that a tournament leads to first-best efforts and truthful ability-revelation if comparisons are across hierarchy levels and not within. The contribution of the current paper is to show that a tournament within a hierarchy level may lead to first-best efforts if prizes are appropriately designed. In contrast to Bhattacharya and Guasch (1988), however, we need the stronger assumption that abilities are common knowledge.

There are also empirical studies that analyze incentives in tournaments with heterogeneous players. Most of these papers conduct experiments so that player types can be assigned to individuals. Their findings point out that disadvantaged individuals often exert more than equilibrium efforts (Bull et al. 1987, Harbring and Lünser 2008). Lynch (2005) uses field data from horse racing to investigate unfair contests. His empirical findings confirm the theoretical result that incentives are large (small) if the degree of heterogeneity between the players is rather small (large).

The paper is organized as follows. Section 2 introduces the model. In Section 3, we derive the optimal tournament contract under uniform prizes. The case of individual prizes is addressed in Section 4. In Subsection 4.1 the employer must satisfy a self-commitment constraint. The optimal tournament contract without this constraint is derived in Subsection 4.2. Section 5 concludes.
2 The Model

Two risk neutral workers $A$ and $B$ are hired by a risk neutral employer $E$. The two workers have reservation values $\bar{u}_A$ and $\bar{u}_B$. Unless stated otherwise, these values are normalized to zero. The workers exert efforts $e_A$ and $e_B$ that lead to monetary output $e_A + e_B$ for the employer. $E$ cannot directly observe efforts nor output but receives an unverifiable, ordinal performance signal $s$ about the ranking of the two workers. This signal either equals $s = s_A$ indicating that worker $A$ has performed better than $B$ or $s = s_B$ indicating the opposite.\(^4\) The signal structure can be characterized as follows:\(^5\)

$$s = \begin{cases} 
  s_A & \text{if } e_A - e_B > \varepsilon \\
  s_B & \text{if } e_A - e_B < \varepsilon.
\end{cases}$$  \hspace{1cm} (1)

Hence, the realization of the relative performance signal depends on the workers’ efforts and the variable $\varepsilon$ that describes an unobservable, exogenous random term (e.g., measurement error) with density $g(\varepsilon)$ and cumulative distribution function $G(\varepsilon)$. The density $g(\varepsilon)$ is assumed to be unimodal and symmetric around zero.\(^6\) Intuitively, the higher worker $i$’s effort choice the more likely the employer will receive the signal $s = s_i$.

Effort $e_i$ entails costs on worker $i$ $(i = A, B)$ that are described in monetary terms by the function $c_i(e_i)$ with $c_i'(e_i), c_i''(e_i), c_i'''(e_i) > 0, \forall e_i > 0$, and $c_i'(0) = c_i(0) = 0$ $(i = A, B)$. We assume that workers differ in ability and, therefore, have different cost functions. Let w.l.o.g. worker $A$ be the more able one with a flatter cost function than his co-worker $B$: $c_A(e) < c_B(e)$, $c_A'(e) < c_B'(e)$, and $c_A''(e) < c_B''(e), \forall e > 0$. Individual cost functions are common knowledge.

\(^4\)Binary signals are often used in principal-agent models. See, among many others, Demougin and Fluet (2001).

\(^5\)Note that we use the same kind of additive model as the seminal paper by Lazear and Rosen (1981), which has become standard in labor economics. We only summarize the difference of idiosyncratic noise via the random variable $\varepsilon$.

\(^6\)The assumption of a unimodal distribution is usual in tournament models; see, e.g., Dixit (1987).
E offers an incentive contract to the workers. Because of the signal structure, this contract must be a simple tournament contract where none of the workers can be handicapped. In the following, we will discuss two different designs $D \in \{UP, IP\}$. On the one hand, $E$ can fix uniform prizes so that the tournament winner receives $w_1$ and the loser $w_2 \leq w_1$ irrespective of which individual worker has performed best ($D = UP$). On the other hand, $E$ can choose individual prizes for the winner and the loser of the tournament so that worker $A$ ($B$) receives $w_{1A}$ ($w_{1B}$) if $s = s_A$ ($s = s_B$) but only $w_{2A}$ ($w_{2B}$) if $s = s_B$ ($s = s_A$) with $w_{2i} \leq w_{1i}$ ($D = IP$). Note that under uniform prizes the important self-commitment property of tournaments – emphasized by Malcomson (1984) – immediately applies whereas under individual prizes the employer may be confronted with a credibility problem: if tournament prizes are individually different $E$ may save labor costs by claiming that worker $i$ is the winner although $s = s_j$ ($i, j = A, B; i \neq j$). This credibility problem of the design $D = IP$ will be discussed in detail in the following sections. $E$ maximizes profits $\pi$ that consist of the sum of the workers’ efforts, $e_A + e_B$, minus tournament prizes.

In the following, we will differentiate between two scenarios. The first one does not impose further restrictions on the tournament prizes so that $E$ is allowed to choose arbitrarily negative prizes, for example. In other words, this scenario assumes unlimited liability for all players. In the second scenario, the two workers are protected by limited liability so that tournament prizes are not allowed to become negative. In particular, the employer cannot use negative loser prizes to extract rents from the workers in this scenario.

As a benchmark, we can calculate the workers’ first-best efforts. These efforts maximize $e_A + e_B - c_A(e_A) - c_B(e_B)$ and are (implicitly) described by

$$1 = c_i'(e_i^{FB}).$$

Since worker $B$’s cost function is steeper than that of worker $A$, we have $e_B^{FB} < e_A^{FB}$.

Finally, to ensure existence of pure-strategy equilibria in the tournament
we assume that\textsuperscript{7}

\[
\sup_{\Delta e} \Delta w \cdot |g'(\Delta e)| < \inf_{\epsilon > 0} c''_A(\epsilon)
\]

with $\Delta e := e_A - e_B$, and $\Delta w$ denoting the spread between winner and loser prize under the respective tournament design.

### 3 Tournaments with Uniform Prizes

We start by considering the standard case of uniform tournament prizes ($D = UP$). Here, the total wage bill is fixed and Malcomson’s self-commitment property of tournaments with unverifiable signals holds. The optimal tournament contract $(w_A^*, w_B^*)$ results from solving a two-stage game where $E$ fixes the tournament prizes at stage 1 and workers observe the prizes and simultaneously choose efforts at stage 2.

We work backwards, starting with the tournament competition. According to (1), the winning-probabilities of worker $A$ and $B$ are given by $G(e_A - e_B)$ and $1 - G(e_A - e_B)$, respectively. The two workers thus maximize

\[
EU_A(e_A) = w_2 + \Delta w G(e_A - e_B) - c_A(e_A)
\]

and

\[
EU_B(e_B) = w_2 + \Delta w [1 - G(e_A - e_B)] - c_B(e_B)
\]

with $\Delta w := w_1 - w_2$ denoting the prize spread. The equilibrium $(e_A^*, e_B^*)$ is described by the first-order conditions\textsuperscript{8}

\[
\Delta wg(e_A^* - e_B^*) = c_A(e_A^*)
\]

and

\[
\Delta wg(e_A^* - e_B^*) = c_B(e_B^*).\]

Since the left-hand sides of equations (4) and (5) are identical but worker $B$ has a steeper cost function than $A$, the equilibrium $(e_A^*, e_B^*)$ is asymmetric.

\textsuperscript{7}For a similar condition see Schöttner (2007).

\textsuperscript{8}Recall that (3) guarantees existence.
This is intuitive. Because of his cost advantage worker $A$ chooses higher effort than $B$: $e_A^* > e_B^*$.

At stage 1, $E$ anticipates the workers’ behavior characterized by (4) and (5) and chooses optimal prizes that maximize $\pi(w_1, w_2) = e_A^* + e_B^* - w_1 - w_2$. As the following proposition shows, in general efficient efforts are not implemented independent of whether or not workers are protected by limited liability.

**Proposition 1** Under $D = \text{UP}$, the employer typically does not implement first-best efforts regardless of whether there are any wealth constraints.

**Proof.** See the Appendix.

With heterogeneous workers and uniform prizes, the employer always leaves a rent to the more able worker $A$ and depending on the liability assumption also to worker $B$. Because of these rents, $E$ faces a trade-off. On the one hand, he wants to maximize surplus, on the other hand, he wants to minimize workers’ rents. This trade-off leads $E$ away from the efficient solution. Interestingly, the implemented effort is not necessarily inefficiently low. To see this, we consider an example with cost functions given by $c_A(e_A) = c(e_A)$ and $c_B(e_B) = c(e_B + \Delta)$ with $\Delta > 0$.

**Corollary 1** Let the workers be protected by limited liability and costs given by $c_A(e_A) = c(e_A)$ and $c_B(e_B) = c(e_B + \Delta)$. Then, equilibrium efforts under $D = \text{UP}$ are implicitly described by $c''(e_i^*) = 2g(\Delta)$. There exists a cut-off value $\bar{g}$ so that $E$ implements inefficiently high effort for both workers if and only if $g(\Delta) > \bar{g}$.

**Proof.** See the Appendix.

As mentioned before, $E$ deviates from the efficient solution to reduce workers’ rents. If workers’ rents are decreasing in the strength of incentives, $E$ may find it beneficial to increase the strength of incentives and, hence, the workers’ efforts above the efficient level. Under limited liability, the strength

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$^9$Note that here $c_B'(0) > 0$.  

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of incentives entirely depends on the size of the winner prize. A higher winner prize in turn affects the workers’ rents in three ways. First, holding both effort levels constant, it naturally raises workers’ rents since the wage payments to the workers increase. Second, holding the opponent’s effort constant, a worker chooses a higher effort\(^{10}\) which further increases his rent as otherwise the worker would have stuck to the initial effort. Finally, holding one’s own effort constant, a worker’s rent is reduced since his opponent increases his effort. If the third effect dominates, the workers’ rents are indeed decreasing in the strength of incentives and \(E\) may gain from implementing an inefficiently high effort.

Formally, worker \(B\)’s rent in equilibrium can be written as\(^{11}\)

\[
R_B \equiv EU_B(e_B^*) = \frac{c_B'(e_B^*)}{g(\Delta)} [1 - G(\Delta)] - c_B(e_B^*).
\]

Differentiating with respect to the implemented effort level gives

\[
\frac{\partial R_B}{\partial e_B^*} = \frac{c_B''(e_B^*)}{g(\Delta)} [1 - G(\Delta)] - c_B'(e_B^*).
\]

If \(E\) induces more than first-best effort, we will have \(c_B'(e_B^*) > 1\). Note that

\[
\frac{c_B''(e_B^*)}{g(\Delta)} [1 - G(\Delta)] < 1 \iff c_B''(e_B^*) < \frac{g(\Delta)}{1 - G(\Delta)}
\]

is true since \(c_B''(e_B^*) = 2g(\Delta)\) and \(G(\Delta) \geq \frac{1}{2}\) due to the symmetry of the distribution. Altogether, this implies \(\partial R_B/\partial e_B^* < 0\). This formalizes our argument. If the workers’ rents are decreasing in effort, it may be optimal to implement more than first-best effort.

It is important to emphasize that the third of the mentioned effects cannot appear in the standard single-agent hidden-action model (or in individual compensation schemes that are frequently used in practice like piece rates). In the basic hidden-action model, the agent’s (worker’s) rent always increases

\(^{10}\)See (4) and (5).

\(^{11}\)The argumentation for worker \(A\) proceeds analogously.
in the strength of incentives if the limited-liability constraint binds so that the principal (employer) never implements an inefficiently high effort. Altogether, this aspect points to a possible advantage of tournaments that has not yet been mentioned in the literature: if workers are protected by limited liability, the employer may prefer tournament competition in order to reduce workers’ rents.

Note that $\Delta$ can be interpreted as a measure of heterogeneity between the workers. A larger $\Delta$ corresponds to more heterogeneous workers. Since $g(\cdot)$ has a unique mode at zero, $g(\Delta)$ will tend to zero if $\Delta$ goes to infinity. Accordingly, efforts may only be inefficiently high if workers are not too heterogeneous. Intuitively, if heterogeneity between the two workers is not too large and hence competition not too uneven, incentives will be quite high for a given prize spread as $\Delta$ determines the workers’ marginal winning probability in equilibrium, $g(\Delta)$ (see equations (4) and (5)). In this case, creating incentives is not too expensive for the employer.

By implementing inefficiently high efforts, the employer generates a kind of rat race between both workers. Such a rat race can be observed in an adverse selection model (Lazear and Rosen 1981, pp. 858–861) or in the presence of career concerns (Holmström 1999). However, in our model the employer induces overinvestment in effort in order to reduce a worker’s rent, which has not been addressed so far.

4 Tournaments with Individual Prizes

In this section, we consider the tournament design $D = IP$ where both prizes may depend on the identity of winner and loser. Note that the identity of the declared winner is verifiable, but $E$ can misrepresent the unverifiable performance signal $s$ to save labor costs. This problem could be eliminated if the sum of winner prize and loser prize is the same irrespective of who is declared winner of the tournament. In the following, we will derive the op-
timal tournament contract under this additional self-commitment constraint and without the constraint.

4.1 Optimal Tournament Contract with Self-Commitment Constraint

If the sum of loser and winner prize is fixed, we have the condition

\[ w_{1A} + w_{2B} = w_{1B} + w_{2A} \]  \hspace{1cm} (6)

as the employer’s self-commitment constraint. Note that this can be transformed into \( w_{1A} - w_{2A} = w_{1B} - w_{2B} =: \Delta \hat{w} \). Although absolute prizes may differ between contestants, the fixing of the total payroll implies that the prize spread is the same for \( A \) and \( B \). This means that both workers still have the same incentive to win the tournament.

At stage 1, \( E \) maximizes \( \hat{\pi} = \hat{e}_A + \hat{e}_B - w_{1A} - w_{2B} = \hat{e}_A + \hat{e}_B - \Delta \hat{w} - w_{2A} - w_{2B} \) by implementing optimal effort levels \( \hat{e}_A \) and \( \hat{e}_B \). Again, we start with the case of unlimited liability. The employer has to consider the workers’ incentive constraints and their participation constraints. In the case of uniform prizes and unlimited liability, only worker \( B \)’s participation constraint was relevant. Now both participation constraints matter since wages for the workers may differ. We obtain the following result:

**Proposition 2** Let the self-commitment constraint be imposed and the workers be unlimitedly liable. Then, under the optimal contract with individual prizes, both workers’ participation constraints are binding and the employer implements first-best efforts \( \hat{e}_A^* = e_A^{FB} \) and \( \hat{e}_B^* = e_B^{FB} \). This result also holds for different reservation values \( \bar{u}_A \neq \bar{u}_B \) of the workers.

**Proof.** See the Appendix. ■

Proposition 2 shows that \( E \) will implement the first-best solution by using individual prizes even if the total prize sum is fixed and handicaps are not feasible. This differs strongly from the findings in Proposition 1 where the
first-best solution was not attainable. The intuition behind Proposition 2 is that \( E \) can fully extract the rent of worker \( A \) by reducing both his loser and his winner prize without violating the self-commitment constraint. Then, \( E \) receives the complete surplus to be produced and implements the efforts that maximize this surplus, i.e. \( e_i^{FB} \).

Now, we turn to the case of \textit{limited liability}. In this scenario, individual prizes do not perform better than uniform ones:

\textbf{Proposition 3} \textit{Let the self-commitment constraint be imposed and the workers be limitedly liable. Then, the optimal contract with individual prizes and the optimal contract with uniform prizes coincide.}

\textbf{Proof.} See the Appendix. \blacksquare

Proposition 3 shows that individual prizes lead to the same outcome as uniform ones if workers are protected by limited liability. Here, \( E \) cannot reduce worker \( A \)’s wages since lowering the loser prize is prevented by the worker’s limited wealth. Accordingly, the employer is not able to extract even a small part of \( A \)’s rent. Hence, the employer will be indifferent between uniform and individual prizes as both tournament designs yield exactly the same (inefficient) effort level.

To summarize the findings of this subsection, individual prizes that satisfy the self-commitment property (6) (weakly) dominate uniform prizes from the employer’s viewpoint. The self-commitment property implies that under tournament design \( D = IP \) each worker faces the same prize spread so that individual prizes cannot be used to make the tournament competition less uneven by using individually adjusted incentives. In other words, because of the additional condition (6) individual tournament prizes cannot serve as a substitute for handicaps, which were impossible in the given setting. Nevertheless, the employer prefers the design \( D = IP \) to \( D = UP \) since individual prizes will be useful to extract rents from the workers if they are not protected by limited liability.
4.2 Optimal Tournament Contract without Self-Commitment Constraint

In this subsection, we relax the restriction (6) on the total payroll, which is possible if the ordinal performance signal is verifiable.\textsuperscript{13} This implies that the prize spread may now differ between contestants. Again, we differentiate between unlimited and limited liability of the workers. Building on the analysis of the previous subsection, the former case is straightforward to solve: we have seen that the first-best solution is implemented if the workers are not protected by limited liability. Of course, this result continues to hold since we have removed a constraint from the optimization problem and, hence, make it easier to achieve the efficient solution.

The solution to the latter case of limited liability, which will be discussed in the remainder of this subsection, is not straightforward. Without self-commitment constraint (6), each worker $i$ faces an individual prize spread $\Delta w_i := w_{1i} - w_{2i}$ where $\Delta w_A$ may be different from $\Delta w_B$. At stage 2, the workers now maximize

\[
EU_A(e_A) = w_{2A} + \Delta w_A G(e_A - e_B) - c_A(e_A)
\]

and

\[
EU_B(e_B) = w_{2B} + \Delta w_B [1 - G(e_A - e_B)] - c_B(e_B)
\]

leading to the first-order conditions that characterize the equilibrium $(e^*_A, e^*_B)$:

\[
F_1 := \Delta w_A g(e^*_A - e^*_B) - c'_A(e^*_A) = 0
\]

and

\[
F_2 := \Delta w_B g(e^*_A - e^*_B) - c'_B(e^*_B) = 0.
\]

In the following, we will analyze the workers’ interaction with respect to changes in the two prize spreads $\Delta w_A$ and $\Delta w_B$. By implicitly differentiating

\textsuperscript{13}Otherwise, $E$ would ex post always annouce that worker ranking that corresponds to the lower sum of winner and loser prizes.
the system of equations (7) and (8) we obtain

\[ |J| = \begin{vmatrix} \frac{\partial F_1}{\partial e_A} & \frac{\partial F_1}{\partial e_B} \\ \frac{\partial F_2}{\partial e_A} & \frac{\partial F_2}{\partial e_B} \end{vmatrix} = \begin{vmatrix} EU''_A (e_A^*) & -\Delta w_A g' (e_A^* - e_B^*) \\ \Delta w_B g' (e_A^* - e_B^*) & EU''_B (e_B^*) \end{vmatrix} \]

\[ = EU''_A (e_A^*) \cdot EU''_B (e_B^*) + \Delta w_A \Delta w_B [g' (e_A^* - e_B^*)]^2 > 0 \]

for the Jacobian determinant with \( EU''_A (e_A^*) = \Delta w_A g' (e_A^* - e_B^*) - c''_A (e_A^*) < 0 \) and \( EU''_B (e_B^*) = -\Delta w_B g' (e_A^* - e_B^*) - c''_B (e_B^*) < 0 \) due to (3), and

\[ \frac{\partial e_A^*}{\partial \Delta w_A} = 1 \left| \begin{array}{cc} -\frac{\partial F_1}{\partial e_A} & \frac{\partial F_1}{\partial e_B} \\ -\frac{\partial F_2}{\partial e_A} & \frac{\partial F_2}{\partial e_B} \end{array} \right| = -g \cdot \frac{(-\Delta w_B g' - c''_B (e_B^*))}{|J|} > 0 \]

\[ \frac{\partial e_A^*}{\partial \Delta w_B} = 1 \left| \begin{array}{cc} -\frac{\partial F_1}{\partial e_A} & \frac{\partial F_1}{\partial e_B} \\ -\frac{\partial F_2}{\partial e_A} & \frac{\partial F_2}{\partial e_B} \end{array} \right| = -\frac{\Delta w_A \cdot g'}{|J|} \]

\[ \frac{\partial e_B^*}{\partial \Delta w_A} = 1 \left| \begin{array}{cc} \frac{\partial F_1}{\partial e_A} & -\frac{\partial F_1}{\partial e_B} \\ \frac{\partial F_2}{\partial e_A} & -\frac{\partial F_2}{\partial e_B} \end{array} \right| = \frac{\Delta w_B \cdot g'}{|J|} \]

\[ \frac{\partial e_B^*}{\partial \Delta w_B} = 1 \left| \begin{array}{cc} \frac{\partial F_1}{\partial e_A} & -\frac{\partial F_1}{\partial e_B} \\ \frac{\partial F_2}{\partial e_A} & -\frac{\partial F_2}{\partial e_B} \end{array} \right| = -\frac{\Delta w_A g' - c''_A (e_A^*)}{|J|} > 0 \]

for the comparative statics with \( g := g (e_A^* - e_B^*) \) and \( g' := g'(e_A^* - e_B^*) \). Therefore, each worker’s equilibrium effort increases in his own prize spread whereas a worker’s reaction to an increase in his opponent’s prize spread depends on whether this increase makes competition more or less even. Consider, for example, the case of \( \partial e_A / \partial \Delta w_B \). If \( e_A > e_B \) we are at the right-hand side of the probability distribution where we have \( g' < 0 \) because of \( g' \)'s unique mode at zero. In this situation, initially worker \( A \) is the stronger player. If now worker \( B \)'s prize spread increases, competition will become more even. Consequently, \( B \) increases his effort and \( A \) increases his effort as well: \( \partial e_A / \partial \Delta w_B > 0 \). Technically, the sign of \( g' \) determines whether reaction functions are upward or downward sloping and, thus, whether efforts are strategic complements or substitutes.
At stage 1, $E$ maximizes his expected profit

$$
\pi = e_A^* + e_B^* - G(e_A^* - e_B^*)(w_{1A} + w_{2B}) - [1 - G(e_A^* - e_B^*)](w_{1B} + w_{2A})
= e_A^* + e_B^* + (\Delta w_B - \Delta w_A) G(e_A^* - e_B^*) - w_{1B} - w_{2A}.
$$

subject to the limited-liability constraints.\(^{14}\) Recall that $e_i^* = e_i^* (\Delta w_i, \Delta w_j)$. When deriving the optimal tournament prizes, first consider $w_{1A}$ and $w_{2A}$. In the optimum, we must have that $\partial \pi / \partial w_{1A} \leq 0$ and $\partial \pi / \partial w_{2A} \leq 0$. Since $\partial \Delta w_A / \partial w_{1A} = -\partial \Delta w_A / \partial w_{2A}$, from $E$’s objective function we obtain

$$
\left. \frac{\partial \pi}{\partial w_{2A}} = - \frac{\partial \pi}{\partial w_{1A}} - 1. \right.
$$

$E$ will always choose an interior solution for the winner prize (that is $w_{1A}^* > 0$) since zero incentives cannot be optimal because of $c_t'(0) = 0$. Hence, $\partial \pi / \partial w_{1A} = 0$, which implies $\partial \pi / \partial w_{2A} = -1 < 0$ and, therefore, a corner solution for the loser prize: $w_{2A}^* = 0$. Analogously, we get $w_{2B}^* = 0$ and $w_{1B}^* > 0$. This is intuitive. If $E$ wants to lower the incentives of a worker, it is always cheaper to decrease the winner prize than to increase the loser prize. Therefore, the two loser prizes are set equal to the lowest possible level.

By inserting $w_{2A}^* = w_{2B}^* = 0$, $E$’s objective function boils down to

$$
\pi = e_A^* + e_B^* + (w_{1B} - w_{1A}) G(\Delta e^*) - w_{1B} \quad \text{where } \Delta e^* := e_A^* - e_B^*
$$

with $\frac{\partial \Delta e^*}{\partial w_{1A}} = \frac{g \cdot c''_B(e_B^*)}{|J|} > 0$ and $\frac{\partial \Delta e^*}{\partial w_{1B}} = -\frac{g \cdot c''_A(e_A^*)}{|J|} < 0$.

The first-order conditions for the optimal winner prizes are

$$
\frac{\partial \pi}{\partial w_{1A}} = \frac{\partial e_A^*}{\partial w_{1A}} + \frac{\partial e_B^*}{\partial w_{1A}} + (w_{1B} - w_{1A}) \frac{\partial \Delta e^*}{\partial w_{1A}} \cdot g - G = 0
$$

$$
\frac{\partial \pi}{\partial w_{1B}} = \frac{\partial e_A^*}{\partial w_{1B}} + \frac{\partial e_B^*}{\partial w_{1B}} + (w_{1B} - w_{1A}) \frac{\partial \Delta e^*}{\partial w_{1B}} \cdot g - (1 - G) = 0
$$

\(^{14}\)Again, we can ignore the workers’ participation constraints.
with $G := G(\Delta e^*)$. These conditions can be simplified to

$$
\frac{\partial \pi}{\partial w_{1A}} = g \cdot \left( \frac{2w_1Bg' + c_B'' (e_B) (1 + (w_{1B} - w_{1A}) g)}{|J|} - \frac{G}{g} \right) = 0
$$

$$
\frac{\partial \pi}{\partial w_{1B}} = g \cdot \left( \frac{-2w_1Ag' + c_A'' (e_A) (1 - (w_{1B} - w_{1A}) g)}{|J|} - \frac{1 - G}{g} \right) = 0.
$$

Defining $k := \sup \{c_B'' (e) - c_A'' (e)\}$ and using the last two equations, we obtain the following result:

**Proposition 4** If workers are protected by limited liability but $E$ does not have to consider the self-commitment constraint, the employer implements $e^*_A \geq e^*_B$. Optimal loser prizes are $w^*_2 = w^*_2 = 0$. Moreover, if $k$ is not too high, optimal winner prizes satisfy $w^*_1 > w^*_1 > 0$.

**Proof.** The claim $w^*_2 = w^*_2 = 0$ has already been proven. To prove that $w^*_1 > w^*_1 > 0$ it suffices to show that

$$
\frac{\partial \pi}{\partial w_{1A}} \bigg|_{w_{1A}=w_{1B}} > \frac{\partial \pi}{\partial w_{1A}} \bigg|_{w_{1A}=w_{1B}}, \forall w_{1A} = w_{1B} \tag{9}
$$

For $w_{1A} = w_{1B} =: w$ we have

$$
\frac{\partial \pi}{\partial w_{1A}} \bigg|_{w_{1A}=w_{1B}} = g \cdot \left( \frac{2wg' + c_B'' (e_B)}{|J|} - \frac{G}{g} \right)
$$

$$
\frac{\partial \pi}{\partial w_{1B}} \bigg|_{w_{1A}=w_{1B}} = g \cdot \left( \frac{-2wg' + c_A'' (e_A)}{|J|} - \frac{1 - G}{g} \right).
$$

From (7) and (8) we know that $w_{1A} = w_{1B}$ leads to $e_A > e_B$ due to $B$’s steeper cost function. This implies $g' < 0$ and $G > \frac{1}{2}$ since the density $g(\cdot)$ is symmetric and unimodal with peak at zero. Hence, if $c_B'' (e) - c_A'' (e)$ is not too large, the inequality (9) is satisfied.

Finally, we have to show that $e^*_A \geq e^*_B$. The proof is by contradiction. Suppose therefore $e^*_A < e^*_B$. From (7) and (8) and $w^*_2 = 0$ it is easy
to see that total wage costs are

\[ G \cdot w^*_1 + (1 - G) \cdot w^*_B = \frac{Gc'_A(e^*_A)}{g(e^*_A - e^*_B)} + \frac{(1 - G) c'_B(e^*_B)}{g(e^*_A - e^*_B)}. \]

Now, let \( E \) change prizes such that efforts \( \hat{e}_A = e^*_B \) and \( \hat{e}_B = e^*_A \) are implemented. This leaves the employer’s output unaffected, while wage costs change to

\[ (1 - G) \hat{w}_1 + G\hat{w}_B = \frac{(1 - G) c'_A(e^*_B)}{g(e^*_B - e^*_A)} + \frac{Gc'_B(e^*_A)}{g(e^*_B - e^*_A)}. \]

The wage difference can be written as

\[ Gw^*_1 + (1 - G) w^*_B - [(1 - G) \hat{w}_1 + G\hat{w}_B] = \]

\[ \frac{Gc'_A(e^*_A)}{g(e^*_A - e^*_B)} + \frac{(1 - G) c'_B(e^*_B)}{g(e^*_A - e^*_B)} - \left[ \frac{(1 - G) c'_A(e^*_B)}{g(e^*_B - e^*_A)} + \frac{Gc'_B(e^*_A)}{g(e^*_B - e^*_A)} \right]. \]

By symmetry of \( g(\cdot) \), we obtain \( g(e^*_A - e^*_B) = g(e^*_B - e^*_A) \). Therefore, the wage difference is positive iff

\[ Gc'_A(e^*_A) + (1 - G) c'_B(e^*_B) - [(1 - G) c'_A(e^*_B) + Gc'_B(e^*_A)] > 0 \]

\( \Leftrightarrow (1 - G) [c'_B(e^*_B) - c'_A(e^*_B)] - G [c'_B(e^*_A) - c'_A(e^*_A)] > 0. \]

The last condition can be transformed into

\[ (1 - 2G) [c'_B(e^*_B) - c'_A(e^*_B)] + G [c'_B(e^*_A) - c'_A(e^*_A) - c'_A(e^*_B) + c'_A(e^*_A)] > 0 \]

or

\[ (1 - 2G) \cdot [c'_B(e^*_B) - c'_A(e^*_B)] + G \cdot \left( \int_{e^*_A}^{e^*_B} c''_B(t) - c''_A(t) \, dt \right) > 0 \]

Because of \( c'_A(e) < c'_B(e) \) and \( c'_A(e) < c'_B(e), \forall e > 0 \), as well as \( G(e^*_A - e^*_B) < 0.5 \) for \( e^*_A < e^*_B \), the last condition is always fulfilled. The employer therefore benefits from implementing efforts \( \hat{e}_A \) and \( \hat{e}_B \) contradicting the assumption that the initial situation was an equilibrium. ■
The intuition for the result of Proposition 4 is the following: the smaller the effort difference $e_A - e_B > 0$ the larger will be the workers’ marginal winning probability $g(e_A - e_B)$ and, hence, the larger will be overall efforts because competition is more even.$^{15}$ Choosing $w_{1B}^* > w_{1A}^*$ exactly serves this purpose. Therefore, without self-commitment constraint, individual tournament prizes are used by $E$ as a substitute for handicaps to make competition more even.$^{16}$ Clearly, the employer would never choose $w_{1B}^*$ so large that worker $B$ becomes the more aggressive contestant because $E$ can induce the same output at smaller wage costs if $e_A > e_B$.

To sum up, the analysis of the tournament design $D = IP$ has shown that the use of individual tournament prizes has two major advantages for the employer. First, individual prizes can be used to extract rents from the workers when the employer has to satisfy a self-commitment constraint. Second, if this constraint is skipped the employer will further use individual prizes to adjust individual incentives so that competition becomes more balanced.

5 Conclusion

In this paper, we have shown that individual tournament prizes dominate uniform ones. If the employer has to satisfy an additional self-commitment condition, individual prizes will be helpful for extracting rents from the workers. If this condition can be neglected, individual prizes exhibit a further advantage. Now they can be used as a substitute for handicaps when adjusting individual incentives in order to make the tournament competition more even. Our paper does not only add to the theory of rank-order tournaments, but also explains empirical observations on pay variation within hierarchy levels and different tournament prizes for individual athletes in professional sports.

$^{15}$This can be directly seen from (7) and (8).

$^{16}$Notice that the restriction on $k$ in Proposition 4 describes a sufficient condition. It is therefore possible that the employer sets $w_{1B}^* > w_{1A}^*$ even if we relax this condition. For instance, if cost functions are again given by $c_A(e_A) = c(e_A)$ and $c_B(e_B) = c(e_B + \Delta)$ with $\Delta > 0$, we will always have $w_{1B}^* > w_{1A}^*$. 
Appendix

Proof of Proposition 1

We consider unlimited liability and limited liability separately. In the former case $E$ maximizes profits subject to incentive constraints (4) and (5) and the participation constraints. The participation constraint for worker $A$ can be neglected because he earns a positive rent: since $c_A(e) < c_B(e), \forall e > 0$, worker $A$ can always choose the same effort level as $B$ (implying a winning probability of 1/2 for each worker) at lower costs. As we have seen from (4) and (5), $A$ even improves his position by choosing strictly larger effort than $B$ at the optimum. Hence, only $B$'s constraint matters:

$$w_2 + \Delta w \left[1 - G(e_A^* - e_B^*)\right] - c_B(e_B^*) \geq 0.$$  \hspace{1cm} (10)

To solve $E$’s optimization problem we set up the Lagrange function

$$L(e_A^*, e_B^*, w_1, w_2) =$$

$$e_A^* + e_B^* - w_1 - w_2 + \lambda_1 \cdot \left[w_2 + \Delta w \left[1 - G(e_A^* - e_B^*)\right] - c_B(e_B^*)\right]$$

$$+ \lambda_2 \cdot \left[\Delta w g(e_A^* - e_B^*) - c'_A(e_A^*)\right] + \lambda_3 \cdot \left[\Delta w g(e_A^* - e_B^*) - c'_B(e_B^*)\right]$$

with $\lambda_1, \lambda_2$ and $\lambda_3$ as multipliers. The optimality conditions yield

$$\frac{\partial L}{\partial e_A^*} = 1 - \Delta w [\lambda_1 g(e_A^* - e_B^*) - g'(e_A^* - e_B^*) (\lambda_2 + \lambda_3)] - \lambda_2 c''_A(e_A^*) = 0$$

$$\frac{\partial L}{\partial e_B^*} = 1 + \Delta w [\lambda_1 g(e_A^* - e_B^*) - g'(e_A^* - e_B^*) (\lambda_2 + \lambda_3)]$$

$$- \lambda_1 c''_B(e_B^*) - \lambda_3 c''_B(e_B^*) = 0$$

$$\frac{\partial L}{\partial w_1} = -1 + \lambda_1 [1 - G(e_A^* - e_B^*)] + g(e_A^* - e_B^*) (\lambda_2 + \lambda_3) = 0$$

$$\frac{\partial L}{\partial w_2} = -1 + \lambda_1 - \lambda_1 [1 - G(e_A^* - e_B^*)] - g(e_A^* - e_B^*) (\lambda_2 + \lambda_3) = 0.$$  \hspace{1cm} (11-14)

Conditions (13) and (14) together give $\lambda_1 = 2$, indicating that the participation constraint is binding. If $E$ implemented the first-best solution, the
optimal prize difference were given by $\Delta w = \frac{1}{g(e^*_A - e^*_B)}$. Then, the upper three optimality conditions could be rewritten as

$$\frac{\partial L}{\partial e_A} = -1 + \frac{g'}{g} (e^*_A - e^*_B) (\lambda_2 + \lambda_3) - \lambda_2 c''_A (e^*_B) = 0$$

$$\frac{\partial L}{\partial e_B} = 1 - \frac{g'}{g} (e^*_A - e^*_B) (\lambda_2 + \lambda_3) - \lambda_3 c''_B (e^*_B) = 0$$

$$\frac{\partial L}{\partial w_1} = 1 - 2G (e^*_A - e^*_B) + g (e^*_A - e^*_B) (\lambda_2 + \lambda_3) = 0.$$

It is easy to see that we have three conditions, but only two endogenous variables $(\lambda_2, \lambda_3)$. One could solve the first two conditions for $\lambda_2$ and $\lambda_3$ and insert the resulting values into the third condition. Thereby, one would obtain a condition that depends entirely on exogenous variables and is only fulfilled by chance.

We now turn to the case of limited liability. Here, the participation constraints of both workers can be neglected: each worker can ensure himself a non-negative expected utility by accepting the contract and choosing zero effort. But now the employer has to take into account the limited-liability constraint $w_1, w_2 \geq 0$. Since $w_1 \geq w_2$, this constraint reduces to $w_2 \geq 0$ and the Lagrange function is given by $(\mu_1, \mu_2$ and $\mu_3$ denote multipliers)

$$L(e^*_A, e^*_B, w_1, w_2) = e^*_A + e^*_B - w_1 - w_2 + \mu_1 \cdot w_2$$

$$+ \mu_2 \cdot [\Delta wg (e^*_A - e^*_B) - c'_A (e^*_A)] + \mu_3 \cdot [\Delta wg (e^*_A - e^*_B) - c'_B (e^*_B)].$$

The corresponding optimality conditions are

$$\frac{\partial L}{\partial e^*_A} = 1 + \Delta wg' (e^*_A - e^*_B) (\mu_2 + \mu_3) - \mu_2 c''_A (e^*_A) = 0$$

(15)

$$\frac{\partial L}{\partial e^*_B} = 1 - \Delta wg' (e^*_A - e^*_B) (\mu_2 + \mu_3) - \mu_3 c''_B (e^*_B) = 0$$

(16)

$$\frac{\partial L}{\partial w_1} = -1 + g (e^*_A - e^*_B) (\mu_2 + \mu_3) = 0$$

(17)

$$\frac{\partial L}{\partial w_2} = -1 + \mu_1 - g (e^*_A - e^*_B) (\mu_2 + \mu_3) = 0.$$  

(18)
From (17) and (18), we immediately obtain that \( \mu_1 = 2 \). Hence, the limited-liability constraint is binding and \( w_2 = 0 \).

If \( E \) implemented the first-best solution, the optimal winner prize were given by \( w_1 = \frac{1}{g(e_A^B - e_B^B)} \). Then, the upper three optimality conditions could be rewritten as

\[
\begin{align*}
\frac{\partial L}{\partial e_A^*} &= 1 + \frac{g'}{g} \left( e_A^F - e_A^B \right) (\mu_2 + \mu_3) - \mu_2 c_A'' (e_A^B) = 0 \\
\frac{\partial L}{\partial e_B^*} &= 1 - \frac{g'}{g} \left( e_A^F - e_B^F \right) (\mu_2 + \mu_3) - \mu_3 c_B'' (e_B^F) = 0 \\
\frac{\partial L}{\partial w_1} &= -1 + g (e_A^B - e_B^F) (\mu_2 + \mu_3) = 0.
\end{align*}
\]

Again, we have a system of three equations for two unknowns \((\mu_2, \mu_3)\). In analogy to the unlimited-liability case, all three conditions can only be fulfilled by chance.

**Proof of Corollary 1**

Conditions (15) and (16) can be combined to

\[
2 = \mu_2 c_A'' (e_A^*) + \mu_3 c_B'' (e_B^*) . \tag{19}
\]

The cost functions \( c_A(e_A) = c(e_A) \) and \( c_B(e_B) = c(e_B + \Delta) \), together with (4) and (5), imply \( c'(e_A^*) = c'(e_B^* + \Delta) \iff e_A^* = e_B^* + \Delta \iff c''(e_A^*) = c''(e_B^* + \Delta) \iff c_A''(e_A^*) = c_B''(e_B^* + \Delta) \). Then, (19) simplifies to \( 2 = (\mu_2 + \mu_3) c''(e_A^*) \).

Furthermore, (17) can be rewritten as \( \mu_2 + \mu_3 = 1/g (e_A^* - e_B^*) = 1/g (\Delta) \) since \( e_A^* = e_B^* + \Delta \). Altogether, optimal efforts under limited liability are characterized by \( c_A''(e_A^*) = 2g(\Delta) \). Since \( c_A''(\cdot) \) is monotonically increasing and first-best efforts are finite according to (2), there exists a cut-off value \( \bar{g} \) so that \( E \) implements inefficiently high effort for both workers if and only if \( g(\Delta) > \bar{g} \).

**Proof of Proposition 2**

The equilibrium at stage 2 is characterized by the workers’ first-order con-
conditions $\Delta \tilde{w} g (\tilde{e}_A - \tilde{e}_B) = c'_A (\tilde{e}_A)$ and $\Delta \tilde{w} g (\tilde{e}_A - \tilde{e}_B) = c'_B (\tilde{e}_B)$. In addition, the workers’ participation constraints are $w_{2A} + \Delta \tilde{w} G (\tilde{e}_A - \tilde{e}_B) - c_A (\tilde{e}_A) \geq \bar{u}_A$ and $w_{2B} + \Delta \tilde{w} [1 - G (\tilde{e}_A - \tilde{e}_B)] - c_B (\tilde{e}_B) \geq \bar{u}_B$. Differentiating the Lagrangian

$$L(\tilde{e}_A, \tilde{e}_B, \Delta \tilde{w}, w_{2A}, w_{2B}) = \tilde{e}_A + \tilde{e}_B - \Delta \tilde{w} - w_{2A} - w_{2B}$$
$$+ \lambda_1 \cdot [w_{2A} + \Delta \tilde{w} G (\tilde{e}_A - \tilde{e}_B) - c_A (\tilde{e}_A) - \bar{u}_A]$$
$$+ \lambda_2 \cdot [w_{2B} + \Delta \tilde{w} [1 - G (\tilde{e}_A - \tilde{e}_B)] - c_B (\tilde{e}_B) - \bar{u}_B]$$
$$+ \lambda_3 \cdot [\Delta \tilde{w} g (\tilde{e}_A - \tilde{e}_B) - c'_A (\tilde{e}_A)] + \lambda_4 \cdot [\Delta \tilde{w} g (\tilde{e}_A - \tilde{e}_B) - c'_B (\tilde{e}_B)]$$

with respect to $w_{2A}$ and $w_{2B}$, we obtain

$$\frac{\partial L}{\partial w_{2A}} = -1 + \lambda_1 \overset{!}{=} 0 \quad \text{and} \quad \frac{\partial L}{\partial w_{2B}} = -1 + \lambda_2 \overset{!}{=} 0.$$

This means that for each worker the participation constraint must bind and the employer extracts all rents from the workers. Moreover, differentiating the Lagrangian with respect to $\tilde{e}_A$, $\tilde{e}_B$ and $\Delta \tilde{w}$ yields

$$\frac{\partial L}{\partial \tilde{e}_A} = 1 + (\lambda_1 - \lambda_2) \Delta \tilde{w} g (\tilde{e}_A - \tilde{e}_B) + (\lambda_3 + \lambda_4) \Delta \tilde{w} g' (\tilde{e}_A - \tilde{e}_B)$$
$$- \lambda_3 c''_A (\tilde{e}_A) - \lambda_1 c'_A (\tilde{e}_A) \overset{!}{=} 0$$

$$\frac{\partial L}{\partial \tilde{e}_B} = 1 - (\lambda_1 - \lambda_2) \Delta \tilde{w} g (\tilde{e}_A - \tilde{e}_B) - (\lambda_3 + \lambda_4) \Delta \tilde{w} g' (\tilde{e}_A - \tilde{e}_B)$$
$$- \lambda_2 c''_B (\tilde{e}_B) - \lambda_4 c'_B (\tilde{e}_B) \overset{!}{=} 0$$

$$\frac{\partial L}{\partial \Delta \tilde{w}} = -1 + \lambda_1 G (\tilde{e}_A - \tilde{e}_B) + \lambda_2 [1 - G (\tilde{e}_A - \tilde{e}_B)] + (\lambda_3 + \lambda_4) g (\tilde{e}_A - \tilde{e}_B) \overset{!}{=} 0.$$

Using $\lambda_1 = \lambda_2 = 1$, the three conditions boil down to

$$1 + (\lambda_3 + \lambda_4) \Delta \tilde{w} g' (\tilde{e}_A - \tilde{e}_B) - \lambda_3 c''_A (\tilde{e}_A) - c'_A (\tilde{e}_A) = 0$$
\( 1 - (\lambda_3 + \lambda_4) \Delta \hat{u} g' (\hat{e}_A - \hat{e}_B) - c'_B (\hat{e}_B) - \lambda_4 c''_B (\hat{e}_B) = 0 \)
\[
\lambda_3 + \lambda_4 = 0,
\]
which together yields
\[
\lambda_3 + \lambda_4 = \frac{1 - c'_A (\hat{e}_A)}{c''_A (\hat{e}_A)} + \frac{1 - c'_B (\hat{e}_B)}{c''_B (\hat{e}_B)} = 0.
\]

As \( c'_A (\hat{e}_A) = c'_B (\hat{e}_B) \), the only solution to this equation has \( c'_A (\hat{e}_A) = c'_B (\hat{e}_B) = 1 \) and, hence, \( \hat{e}_A^* = e_{FB}^A \) and \( \hat{e}_B^* = e_{FB}^B \). The corresponding tournament prizes are given by
\[
\begin{align*}
    w_{2A} &= \bar{u}_A + c_A (e_{FB}^A) - \frac{G (e_{FB}^A - e_{FB}^B)}{g (e_{FB}^A - e_{FB}^B)} \\
    w_{1A} &= \bar{u}_A + c_A (e_{FB}^A) + \frac{1 - G (e_{FB}^A - e_{FB}^B)}{g (e_{FB}^A - e_{FB}^B)} \\
    w_{2B} &= \bar{u}_B + c_B (e_{FB}^B) - \frac{1 - G (e_{FB}^B - e_{FB}^A)}{g (e_{FB}^B - e_{FB}^A)} \\
    w_{1B} &= \bar{u}_B + c_B (e_{FB}^B) + \frac{G (e_{FB}^B - e_{FB}^A)}{g (e_{FB}^B - e_{FB}^A)}.
\end{align*}
\]

**Proof of Proposition 3**

Under limited liability and individual prizes, we can again neglect the workers’ participation constraints as for the case of uniform prizes. Instead, the employer now faces the two limited-liability conditions \( w_{2A} \geq 0 \) and \( w_{2B} \geq 0 \). Because of the non-binding participation constraints and the fact that loser prizes decrease incentives and increase labor costs, the employer optimally chooses \( w_{2A} = w_{2B} = 0 \). But then (6) implies \( w_{1A} = w_{1B} \) and the optimization problem for individual prizes becomes identical to that for uniform prizes.
References


